Toroidal Cavity as an Accelerator Module[1]

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Abstract

In this work we have developed the successive aproximation method (SAM) to determine electromagnetic fields in toroidal cavities. The method is tested on the example of a coaxial cylinder cavity excited on toroidal modes. The method's correctness is studied and the accuracy of determination of own frequencies and amplitudes of electromagnetic oscillations is estimated for the modes interesting from the viewpoint of particle acceleration.

1. INTRODUCTION

The problem of determination of proper electromagnetic oscillations in toroidal cavities just like in some other compound cavities consists in the fact that in the considered coordinate systems the variables in Maxwell equations cannot be separated. Meanwhile it is of interest to study a total pattern of field with all possible types and modes of oscillations arising in a toroidal cavity at its excitation at a given frequency and type of polarization in terms of constructing an effective accelerative module.

The authors of [1,2] suggested an asymptotic method of variables separation and determination of own electromagnetic oscillations in toroidal cavities. The method is valid for "big" tori. In the present talk this problem is solved by the SAM [3]. To justify the correctness of the formulated problem and to control the accuracy of obtained expressions we consider a toroidal cavity with rectangular cross-section - a coaxial cylinder - when there exists a strict solution for coaxial oscillations. Obviously, the same accuracy can be attributed to solutions obtained quite similarly for a genuine tori with a round cross-section.

The method developed in this work allows to study the effect of occurrence of duplet modes due to removal of oscillation degeneracy of torus curvature.

2. METHOD OF SOLUTION

We'll proceed from Maxwell equations written in a "quasicylindrical" coordinate system (z, ξ, φ) connected with the Cartesian system by relations

$$z=z,\quad x+iy=h_{\varphi}e^{i\varphi}$$

where

$$0\le z\le 2b,\quad -1\le\xi\le 1,\quad 0\le\varphi\le 2\pi$$
$$h\varphi=Rh=R(1+\rho_o\xi),\quad \rho_o=a/R \tag{1}$$

If we classify the oscillations in a toroidal cavity relative to the cylinder z axis as E - $(H_z=0)$ and H - $(E_z=0)$ types of waves, we will arrive at strict expressions for coaxial oscillations described by potential functions Φ_z [3]:

$$\Phi_z=\begin{Bmatrix}E_z\\ H_z\end{Bmatrix}=\left\{J_m\left(\chi\frac{1+\rho_o\xi}{1-\rho_o}\right)\begin{Bmatrix}N_m(\chi)\\ N_m'(\chi)\end{Bmatrix}-N_m\left(\chi\frac{1+\rho_o\xi}{1-\rho_o}\right)\begin{Bmatrix}J_m(\chi)\\ J_m'(\chi)\end{Bmatrix}\right\}\cos m\varphi\begin{Bmatrix}\cos\frac{\pi na}{b}\\ \sin\frac{\pi na}{b}\end{Bmatrix} \tag{2}$$

wherein the eigenvalues for two types of wave

$$\chi=\omega a/c=[\chi^{(1,2)}_{mp}+(\pi n/b)^2]^{1/2}$$

are also determined from strict dispersion equations

$$J_m(\chi^{(1)}_{mp})N_m\left(\frac{1+\rho_o\xi}{1-\rho_o}\chi^{(1)}_{mp}\right)=J_m\left(\frac{1+\rho_o\xi}{1-\rho_o}\chi^{(1)}_{mp}\right)N_m(\chi^{(1)}_{mp} \tag{3a}$$

$$J_m'(\chi^{2}_{mp})N_m'\left(\frac{1+\rho_o\xi}{1-\rho_o}\chi^{(2)}_{mp}\right)=J_m'\left(\frac{1+\rho_o\xi}{1-\rho_0}\chi^{(2)}_{mp}\right)N_m(\chi^{(2)}_{mp}) \tag{3b}$$

for E (3a) and H (3b) types of oscillations.

Having chosen as a smallness parameter the torus curvature $\rho=a/R$, we will try to solve the problem using SAM. We are looking for the SAM solutions as an expansion in powers of that parameter

$$\Phi_z 0\begin{Bmatrix}E_z\\ H_z\end{Bmatrix}=\sum_l\begin{Bmatrix}E_l(\xi)\rho_o^l\\ H_l(\xi)\rho_o^l\end{Bmatrix}\cos m\varphi\begin{Bmatrix}\cos\frac{\pi nz}{b}\\ \sin\frac{\pi nz}{b}\end{Bmatrix} \tag{4a}$$

$$\chi^2=\sum_l\chi_l^2\rho_o^2,\qquad \chi^2=\begin{Bmatrix}\chi^{(1)2}_{mpn}\\ \chi^{(2)2}_{mpn}\end{Bmatrix}+\left(\frac{\pi n}{b}\right)^2 \tag{4b}$$

Note that the expansion coefficients in (4b) at each stage of iterations are obtained by terwmise satisfaction of boundary conditions of expansions (4a) for fields. They turn out nonzero

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for even values of l, whereas at odd approximations in (4a) there also participate solutions of homogeneous equations with coefficients determined from the boundary conditions. Comparision of calculations of strict solutions (3a) and (3b) with results of these calculations shows that error in determination of eigenvalues of $\chi^{(1,2)}_{mpn}$ by SAM does not exceed 0.5% for $\rho_o \leq 0.9$ under restriction in expansion (4b) by three terms, i.e. at $0(\rho o^8)$ and 1.5÷3% at accuracy $0(\rho 6 o)$. In that case the occured duplet lines are revealed already at calculation accuracy $(\rho 6_o)$.

Further arguments can be based on the fact that electromagnetic oscillations in cavities are twice degenerate [3,4,5], i.e. in the coaxial oscillations stripulated by E_z or H_z, there can exist, for example, toroidal oscillations due to E_φ or H_φ. However, as follows from [6,7], such a degeneracy will only take place at $n=0$ or $m=0$. Generally, the independent existence of E_φ and H_φ oscillations in a toroidal cavity can be only spoken of asymptotically for very large tori $(\rho_o \to 0)$. Having certified this and basing on high accuracies provided by SAM, we turn to searching for toroidal oscillations using that method. Having classified electromagnetic fields by E - $(H_\varphi=0)$ and H - $(E_\varphi=0)$ types of waves with respect to the direction of azimuthal angle φ, potential functions Φ_φ in the coordinate system [1] we obtain equations

$$\left(\chi^2 h^2 - m^2\rho_0^2\right)\left[\frac{\partial^2\Phi_\varphi}{\partial\xi^2} + \frac{\rho_0}{h}\frac{\partial\Phi_\varphi}{\partial\xi} + \left(\chi^2 - \upsilon^2\right)\Phi_\varphi\right] =$$

$$= \chi^2(1+m^2)\rho_0^2\Phi_\varphi + \frac{2m^2}{h}\rho_0^3\frac{\partial\Phi_\varphi}{\partial\xi} + \frac{m^2(1-m^2\rho_0^2)}{h^2}\rho_0^2\Phi_\varphi \qquad (5)$$

reduce to equations for the first derivative Bessel function or Bessel function proper at $\nu=0$ or $m=0$, i.e in the cases when self-dependent existance of E or H types of waves is possible.

For toroidal oscillations of "E-type" the solution of equation (5) by SAM is searched for in the form of a power series (for uniform toroidal modes $m=0$)

$$E_\varphi = sin\frac{\pi nz}{2b}\frac{\sum_l E_l(\xi)\rho_0^l}{\sqrt{1+\rho_0\xi}} \qquad (6a)$$

and expansion (4b) with an accuracy up to $0(\rho_o^5)$ is obtained in the form

$$\chi^{(1)2}_{0pn} = \frac{\pi^2}{4}\left(p^2 + \frac{a^2}{b^2}n^2\right) + \frac{3}{4}\rho_0^2 + \frac{3}{4}\left(1 - \frac{6}{\pi^2 p^2}\right)\rho_0^4 \qquad (6b)$$

In (6a) with an accuracy up to $0(\rho_o^5)$we have the following expressions

$$E_{l=0} = \begin{cases} sin(\pi p/2)\xi & \text{when } p \text{ is even} \\ cos(\pi p/2)\xi & \text{when } p \text{ is odd} \end{cases} \qquad (7a)$$

$$E_{l=1} = E_{l=2} = 0 \qquad (7b)$$

$$E_{l=3} = \frac{3}{2}\frac{(\xi^2-1)}{\pi^2 p^2}E'_{l=0} - \frac{3}{2}\frac{\xi}{\pi^2 p^2}E_{l=0} \qquad (7c)$$

$$E_{l=4} = -\frac{3}{2}\frac{\xi(\xi^2-1)}{\pi^2 p^2}E'_{l=0} + \frac{9\xi^2}{4\pi^2 p^2}E_{l=0} \qquad (7d)$$

Equations (6b) and (7) concide with the expressions for the uniform H-modes in (4a) and (4b).

Other field components are obtained from Maxwell equations and are expressed through E_φ by relations (8). Similar expressions are obtained for "H-types" $(E_\varphi=0)$ of oscillations if in (8) we make replacements $\vec{H} \to -\vec{E}, \vec{E} \to \vec{H}$ and substitute the corresponding expressions for H_φ and $\chi^{(2)}_{mnp}$.

$$E_\xi = \frac{\rho_0}{\chi^2_{mnp}h^2 - m^2\rho_0^2}\frac{\partial^2 hE_\varphi}{\partial z\partial\varphi};\ H_\xi = \frac{\chi ha/\omega}{\chi^2 h^2 - m^2\rho_0}\frac{\partial^{2h}E_\varphi}{\partial z\partial}$$

$$E_z = \frac{\rho_0 a}{\chi^2_{mnp}h^2 - m^2\rho_0^2}\frac{\partial^2 hE_\varphi}{\partial z\partial\varphi};\ H_z = -\frac{\chi h/\omega}{\chi^2_{mnp}h^2 - m^2\rho_0^2}\frac{\partial^2 hE_\varphi}{\partial\xi\partial} \qquad (8)$$

As was already stated, it is senseful to speak of E and H types of toroidal oscillations absolutely strictly in terms of electrodynamics if $m=0$, i.e. in case of homogeneous toroidal modes. Indeed, as follows from formulae (8), at $m=0$, $E_\xi=E_z=0$, and E- wave with components $E_\varphi H_\xi H_z$ will excite in the torus, or H-wave with components $H_\varphi E_\xi H_z$.

On the other hand, from the same formulae (8) it follows that components E_ξ and E_z (H_ξ and H_z - for magnetic type of waves) are proportional to curvature ρ_0 which is assumed small. Hence, for sufficiently large tori we can claim that the separation into E and H types of waves for any kind of modes $(m\neq 0)$ can be spoken of approximately, when the cited components become negligibly small, and SAM is effective for determining homogeneous toroidal modes a tany value of $\rho_0 < 1$.

For homogeneous toroidal modes with square cross-section the modes with indexes $0pn$ and $0np$ form duplets which diverge with increasing ρ_0 and at $\rho_0 \to 0$ they correspond to degenerate modes of rectilinear waveguide (degeneracy $np \to pn$). One also can observe removal of E-H degeneracy of modes H_{011} and E_{011}.

All above-stated allows us to study electromagnetic toroidal oscillations in a "genuine" torus with a round cross-section if we consider Maxwell equations in a "quasi-spherical" coordinate system with Lame`metric coefficients $h_r=1$, $h_\theta=r$, $h_\varphi=Rh=R(1-\rho_0\,\xi\,cos\theta)$, where $\rho_0 = a/R$, a is torus radius and $-1<\xi<1$ [3]. As distinct from the coaxial cavity, any strict solution does not exist in this case, and correctness of SAM application is determined by what is stated above. Evidently, in this case, too, the self-dependent existence of E and H types of waves can be spoken of strictly only for homogeneous toroidal modes.

3. Investigation of Toroidal Cavity

With the aim to check experimentally the strictness and correctness of the calculation methods we designed and manufactured a toroidal cavity with rectangular cross-section (coaxial cylinder). That cavity was excited by a radiating slot located along the generatrix. In an exciting rectangular waveguide there propagates wave H_{10} and the waveguide is oriented so that the electric vector turns out normal to the radiating slot.

The types of fields in the cavity were studied using successively radially aranged probes which thanks to their curving and turning by 90° could fix fields $E_r(\varphi)$ and $E_\varphi(\varphi)$ separately.

Modes excited in the cavity at various values of the internal core diameter coincide with calculated ones. The accuracy of coincidence of the measured resonance frequencies with calculated ones is $\cong 1.5\%$. The cavity geometry can easily be varied by replacing the inner rod with some other one, i.e. $a/b, \rho_0$ are varied.

The above performed measurements and calculations show that at the given polarization of the exciting wave the toroidal mode E_{011} will excite in the cavity (and mode H_{011} at coaxial classification). If we choose the cavity geometry so that $a/b=1$, $\rho_0 \geq 0.75$, this mode (H_{011}) at the frequency 2.972 GHz will be pretty far from other excited modes (H_{111}, H_{211}, H_{311} etc.), i.e. from toroidal and coaxial modes.

Experimental study has shown that at cavity's external diameter 175 mm, inside rod diameter 25 mm and height 75 mm the nearest resonanses from H_{011} mode are approximately 150 MHz apart, which fact provides sufficient decoupling and stable operation of the cavity when using it in systems of interacton of charged particles with electromagnetic fields. In particular, such a system is convenient when studying plasma pinches of high density. This system can be used in charged particle acceleration as well thanks to its advantage: at high Q-factor at (high fields strengths). The system is distinguished by high electric reliability, since the rotational electric field has a maximum in the toroids centre being zero at its walls.

4. REFERENCES

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