Time-Dependent Normal-Mode Analysis of the Wakefield Generated in a Cylindrical Cavity by an Accelerated Charged particle Beam

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Abstract
The wakefield driven in a conducting pipe by a coasting, relativistic, charged particle beam has been widely studied, most of the time thanks to computer codes. In the particular case of a cylindrical "pill box" cavity, analytical expressions of the (**E**,**B**)(**x**,*t*) map have been obtained as developments on the complete base of cavity normal modes. We extend this method to the case of an accelerated beam, which leaves the downstream face of the cavity with a thermal velocity, and reaches the opposite face with a relativistic velocity, a situation encountered in photoinjectors. According to the relative values of the accelerating field and of the cavity geometrical parameters -radius, gap- causality allows to introduce various simplifications. Results are compared to those obtained elsewhere [1] by a direct resolution of Mawxell's equations based on transform and Green's function techniques.

1 INTRODUCTION

Wake fields have been widely studied for coasting charged particle beams propagating inside a conducting cavity. In the case of an ultrarelativistic beam, the field directly generated by beam particles in their wake can be neglected, and the so-called wakefield is the electromagnetic linear response of the cavity to the exciting signal which is the beam. For a transrelativistic beam, the direct field must be taken into account and added to cavity response, which is no longer linear, except for a low-intensity beam. Wakefield effects, often limited to the global effect on the beam of the cavity it has crossed, have been studied for various cavity geometries, most of the time thanks to computer codes. In the particular case of a cylindrical "pill box" cavity, analytical expressions of the (**E**,**B**)(**x**,*t*) map have been obtained as developments on the complete base of cavity normal modes.

We extend this method to the case of an accelerated beam, which leaves the downstream face of the cavity with a thermal velocity, and becomes relativistic in a few cm, a situation encountered in photoinjectors, where the accelerating field is the *E*-field of an RF cavity.

In numerical applications, the considered photoinjector will be the first one of the ELSA facility [2] (CEA, Bruyères-le-Châtel), schematized in Fig. 1 of the companion paper [3], which works at 144 MHz. For this frequency, the *E*-field may be considered as constant for beam pulse lenghts of the order of 100 ps or smaller.

2 STARTING EQUATIONS. MODELLING

They are the same as in the companion paper [1] to which we refer the reader for details on initial and boundary conditions. In brief, we solve Maxwell equations for potentials, in Coulomb gauge :
$\Delta\Phi = -\rho/\varepsilon_0$, $\Box\mathbf{A} = \mu_0\mathbf{j} - (1/c^2)(\partial/\partial t)\nabla\Phi$ in the cylindrical cavity $\mathcal{D}$: $0 \le z \le g \,;\, 0 \le r \le \mathcal{R}$. Due to causality prescriptions, this very simple geometry is a good photoinjector model for wakefield studies [3]

The source terms : (H : Heaviside)

$$\rho(r,z,t) = \frac{I\varpi(z,t)}{\pi a^2 \beta(z) c}[1 - H(r-a)],$$

$$\mathbf{j}(r,z,t) = \beta(z) c\, \rho(r,z,t)\, \mathbf{u}_z$$

describe a radially uniform beam (radius a), of velocity $v(z)=\beta(z)c$, with an arbitrary longitudinal current profile $\varpi(z,t)$.

3 MODAL ANALYSIS

The formalism is the same as the one which has been widely used in studying wakefields of coasting beams. Solutions are sought out as developments :

$$\mathbf{A}(\mathbf{x},t) = \sum_\lambda q_\lambda(t)\mathbf{a}(\mathbf{x}), \qquad \Phi(\mathbf{x},t) = \sum_\lambda r_\lambda(t)\varphi_\lambda(\mathbf{x})$$

on the two complete bases formed by :

- the solutions $\varphi_{np}(\mathbf{x})$ of Helmoltz's scalar equation:
$$\Delta\Phi + \kappa_\Phi{}^2\Phi = 0,$$
for the prescribed boundary conditions on Φ

- the solutions $\mathbf{a}_{np}(\mathbf{x})$ of Helmoltz's vector equation
$$\Delta\mathbf{A} + \kappa_A{}^2\mathbf{A} = 0,$$
for the prescribed boundary conditions on **A**.

$\varphi_{np}(\mathbf{x})$ and $\mathbf{a}_{np}(\mathbf{x})$ are well known for the considered $(0,R)\times(0,g)$ cylindrical cavity :

$$\varphi_{n,p}(\mathbf{x}) = \varphi_0 \, J_0(j_n \frac{r}{\mathcal{R}}) \sin(\frac{p\pi z}{g}),$$

$$\mathbf{a}_{n,p}(\mathbf{x}) = a_0 \left\{ \frac{p\pi}{g} J_1(j_n \frac{r}{\mathcal{R}}) \sin(\frac{p\pi z}{g}) \mathbf{u}_r + \frac{j_n}{\mathcal{R}} J_0(j_n \frac{r}{\mathcal{R}}) \cos(\frac{p\pi z}{g}) \mathbf{u}_z \right\}$$

(ϕ_0 et a_0 : arbitrary amplitudes, functions of the injected HF power) ; J : Bessel ; $\mathbf{u}_r, \mathbf{u}_z$: unit radial and axial vectors.

On the other hand :

$$\kappa_\Phi = \kappa_A = \frac{\omega_{np}}{c} = \left[\left(\frac{j_n}{\mathcal{R}} \right)^2 + \left(\frac{p\pi}{g} \right)^2 \right]^{1/2},$$

where $n = 1,2,3,\ldots$; $p = 0,1,2,\ldots$ j_n is the nth zero of J_0.

As for the coordinates $r_\lambda(t)$ of Φ and $q_\lambda(t)$ of $\mathbf{A}$, a classical calculation [4] links them to source terms $\rho(\mathbf{x},t)$ and $\mathbf{j}(\mathbf{x},t)$ by :

$$\ddot{q}_\lambda + \omega_\lambda^2 \, q_\lambda = \frac{1}{2U_\lambda} \int \mathbf{j}.\mathbf{a}_\lambda \, d^3x$$

$$r_\lambda = \frac{1}{2T_\lambda} \int \rho \, \varphi_\lambda \, d^3x,$$

where :

$$U_\lambda = \frac{\varepsilon_0}{2} \int \mathbf{a}_\lambda^2 \, d^3x,$$

$$T_\lambda = \frac{\varepsilon_0}{2} \int (\nabla \varphi_\lambda)^2 \, d^3x \quad .$$

For the considered cylindrical cavity :

$$\omega_{n,p}^2 = (\frac{j_n}{\mathcal{R}})^2 + (\frac{p\pi}{g})^2$$

$$T_{n,p} = \frac{\pi \varepsilon_0 R^2 g}{4c^2} \, \varphi_0^2 \, J_1^2(j_n) [(\frac{j_n}{\mathcal{R}})^2 + (\frac{p\pi}{g})^2]$$

$$U_{n,p} = \frac{\pi \varepsilon_0 R^2 g}{4} \, a_0^2 \, J_1^2(j_n) [(\frac{j_n}{\mathcal{R}})^2 + (\frac{p\pi}{g})^2].$$

4 A BRIEF OUTLINE OF CALCULATIONS. ANALYTICAL RESULTS

4.1. A brief outline of calculations

The generalized coordinates : $r_\lambda(t)$ of Φ and $q_\lambda(t)$ of $\mathbf{A}$, are first calculated from the above quoted formulæ through Green's function methods. Then, from the rebuilt potentials, the longitudinal (irrotational) $\mathbf{E}_\parallel = -\nabla\Phi$, and transverse (solenoidal) $\mathbf{E}_\perp = -\partial \mathbf{A}/\partial t$, components of the electric field, as well as the magnetic field $\mathbf{B}$ are deduced. According to symmetries, the only non-vanishing components, in cylindrical coordinates, are : E_z, E_r, and B_θ.

Analytical expressions of these fields appear as double sums over the radial n, and axial p indices .

For $E_\parallel$, summation over one of both indices is possible owing to the properties of Bessel functions and Dini's series for n, of Fourier series for p. For $E_\perp$ and B_θ, double summation cannot be avoided.

4.2. Analytical results

Though an arbitrary axial current profile can be handled, we restrict the following results, for sake of simplicity, to a beam pulse of time lenght τ, with stiff front- and back- profiles.

Reduced coordinates and quantities will be used, based upon the characteristic lenght $H^{-1} = mc^2/eE_0$, where e and m are the electron charge and mass respectively, and E_0 the RF-electric field amplitude on the photoinjector cathode : $R=Hr$, $Z=Hz$, $\rho=H\mathcal{R}$, $G=Hg$; $T=Hct$, $\mathcal{T}=Hc\tau$.

a) Longitudinal E-field ; summation over p

$$E_{z_\parallel}(R,Z,T) = \frac{-2I}{\pi\varepsilon_0 c\rho a} \sum_{n=1}^{\infty} \frac{J_0(j_n \frac{R}{\rho}) \, J_1(j_n \frac{A}{\rho})}{j_n \, J_1^2(j_n)} \cdot \frac{\pi}{\mathrm{sh}(j_n \frac{G}{\rho})} \cdot$$

$$\left\{ -\mathrm{ch}\left[j_n \frac{G-Z}{\rho} \right] \int_{T-\mathcal{T}}^{\sqrt{Z(Z+2)}} \mathrm{sh}[\frac{j_n}{\rho}(\sqrt{1+u^2}-1)] \, du + \mathrm{ch}\left[j_n \frac{Z}{\rho} \right] \int_{\sqrt{Z(Z+2)}}^{T} \mathrm{sh}[\frac{j_n}{\rho}(G+1-\sqrt{1+u^2})] \, du \right\}$$

$$E_{r_\parallel}(R,Z,T) = \frac{4I}{\pi\varepsilon_0 c a\rho} \sum_{n=1}^{\infty} \frac{J_1(j_n \frac{R}{\rho}) \, J_1(j_n \frac{A}{\rho})}{j_n \, J_1^2(j_n)} \cdot \frac{\pi}{\mathrm{sh}(j_n \frac{G}{\rho})}$$

$$\left\{ \mathrm{sh}\left[j_n \frac{G-Z}{\rho} \right] \int_{0}^{\sqrt{Z(Z+2)}} \mathrm{sh}[\frac{j_n}{\rho}(\sqrt{1+u^2}-1)] \, du + \mathrm{sh}\left[j_n \frac{Z}{\rho} \right] \int_{\sqrt{Z(Z+2)}}^{T} \mathrm{sh}[\frac{j_n}{\rho}(G+1-\sqrt{1+u^2})] \, du \right\}$$

b) Longitudinal E-field ; summation over n

(I,K modified Bessel functions ; $k=\pi/G$)

$$E_{z_\parallel}(R,Z,T) = -\frac{2I}{\pi\varepsilon_0 c A^2 g} \sum_{p=1}^{\infty} \frac{\cos(kpz)}{kp} \cdot \left\{ \int_{\max(0,T-\mathcal{T})}^{T} \sin[kp(\sqrt{1+u^2}-1) \, du \right\}.$$

$$\begin{cases} 1 - Akp \, I_0(Rkp) \left[K_1(Akp) + \dfrac{K_0(\rho kp)}{I_0(\rho kp)} I_1(Akp) \right] & 0 \le R \le A \le \rho \\ Akp \, I_1(Akp) \left[K_0(Rkp) - \dfrac{K_0(\rho kp)}{I_0(\rho kp)} I_0(Rkp) \right] & 0 \le A \le R \le \rho \end{cases}$$

$$E_{r_{\parallel}}(R,Z,T)=\frac{2I}{\pi\varepsilon_0 cAg}\sum_{p=1}^{\infty}\sin(kpz)\ \cdot$$

$$\left\{\int_{\max(0,T-\mathcal{T})}^{T}\sin[kp(\sqrt{1+u^2}-1)\,du\right\}\cdot$$

$$\begin{cases} \mathrm{I}_1(Rkp)\left[\mathrm{K}_1(Akp)+\dfrac{\mathrm{K}_0(\rho kp)}{\mathrm{I}_0(\rho kp)}\mathrm{I}_1(Akp)\right] & 0\le R\le A\le\rho \\ \mathrm{I}_1(Akp)\left[\mathrm{K}_1(Rkp)+\dfrac{\mathrm{K}_0(\rho kp)}{\mathrm{I}_0(\rho kp)}\mathrm{I}_1(Rkp)\right] & 0\le A\le R\le\rho \end{cases}$$

c) Transverse E-field and magnetic field $\mathbf{B}=B_\theta\,\mathbf{u}_\theta$

Expressions differ depending on whether the considered point is inside the beam, upstream or downstream. For a point located inside the beam :

$$E_{z_\perp}(R,Z,T)=\frac{-4I}{\pi\varepsilon_0 cAg\rho}\sum_{n=1}^{\infty}\frac{\mathrm{J}_0(\mathrm{j}_n\frac{R}{\rho})\,\mathrm{J}_1(\mathrm{j}_n\frac{A}{\rho})}{\mathrm{j}_n\,\mathrm{J}_1^2(\mathrm{j}_n)}\ \cdot$$

$$\left\{\frac{1}{2}\left[\int_0^T(\sqrt{1+u^2}-1)\cos[\frac{\mathrm{j}_n}{\rho}(T-u)]du\right.\right.$$

$$\left.-\int_{\mathcal{T}}^{T}(\sqrt{1+(u-\mathcal{T})^2}-1)]\cos[\frac{\mathrm{j}_n}{\rho}(T-u)]\,du\right]$$

$$+\sum_{p=1}^{\infty}\frac{\mathrm{j}_n^2\cos(kpZ)}{kp[\mathrm{j}_n^2+(\rho kp)^2]}\ \cdot$$

$$\left[\int_0^T\cos[\sqrt{\mathrm{j}_n^2+(\rho kp)^2}\,\frac{T-u}{\rho}]\sin[kp(\sqrt{1+u^2}-1)]\,du-\right.$$

$$\left.\left.\int_{\mathcal{T}}^{T}\cos[\sqrt{\mathrm{j}_n^2+(\rho kp)^2}\,\frac{T-u}{\rho}]\sin[kp(\sqrt{1+(u-\mathcal{T})^2}-1)]\,du\right]\right\}$$

$$E_{r_\perp}(R,Z,T)=\frac{-4I}{\pi\varepsilon_0 cAg}\sum_{n=1}^{\infty}\frac{\mathrm{J}_1(\mathrm{j}_n\frac{R}{\rho})\,\mathrm{J}_1(\mathrm{j}_n\frac{A}{\rho})}{\mathrm{J}_1^2(\mathrm{j}_n)}\ \cdot$$

$$\sum_{p=1}^{\infty}\frac{\sin(kpZ)}{\mathrm{j}_n^2+(\rho kp)^2}\ \cdot$$

$$\left\{\int_0^T\cos[\sqrt{\mathrm{j}_n^2+(\rho kp)^2}\,\frac{T-u}{\rho}]\sin[kp(\sqrt{1+u^2}-1)]\,du\ -\right.$$

$$\left.\int_{\mathcal{T}}^{T}\cos[\sqrt{\mathrm{j}_n^2+(\rho kp)^2}\,\frac{T-u}{\rho}]\sin[kp(\sqrt{1+(u-\mathcal{T})^2}-1)]\,du\right\}$$

$$B_\theta(R,Z,T)=\frac{4\mu_0 I}{\pi Ag\rho}\sum_{n=1}^{\infty}\frac{\mathrm{J}_1(\mathrm{j}_n\frac{R}{\rho})\,\mathrm{J}_1(\mathrm{j}_n\frac{A}{\rho})}{\mathrm{j}_n\,\mathrm{J}_1^2(\mathrm{j}_n)}\ \cdot$$

$$\left\{\frac{1}{2}\left[\int_0^T(\sqrt{1+u^2}-1)]\sin[\frac{\mathrm{j}_n}{\rho}(T-u)]du\right.\right.$$

$$\left.-\int_{\mathcal{T}}^{T}(\sqrt{1+(u-\mathcal{T})^2}-1)]\sin[\frac{\mathrm{j}_n}{\rho}(T-u)]du\right]$$

$$\sum_{p=1}^{\infty}\frac{\mathrm{j}_n\cos(kpZ)}{kp\sqrt{\mathrm{j}_n^2+(\rho kp)^2}}\ \cdot$$

$$\left[\int_0^T\sin[\sqrt{\mathrm{j}_n^2+(\rho kp)^2}\,\frac{T-u}{\rho}]\sin[kp(\sqrt{1+u^2}-1)]\,du\right.$$

$$\left.\left.-\int_{\mathcal{T}}^{T}\sin[\sqrt{\mathrm{j}_n^2+(\rho kp)^2}\,\frac{T-u}{\rho}]\sin[kp(\sqrt{1+(u-\mathcal{T})^2}-1)]\,du\right]\right\}$$

5 COMPARISON WITH THE WAKEFIELD EXPRESSIONS OBTAINED IN [1]

The agreement is excellent. As an exemple, Fig. 1 shows the E_z-wakefield on the axis, as a function of $Z=Hz$, for $t=t_g/2$, where t_g is the time at which the beam head reaches the cavity exit.

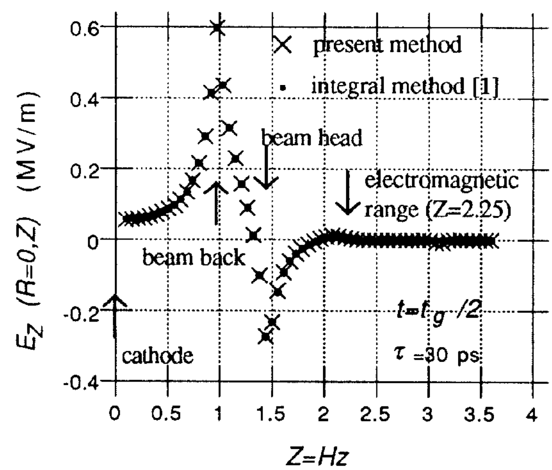

Fig. 1. On an exemple : comparaison between the results of the present method and the one of the integral method [1]

6 REFERENCES

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