Trapped Modes In Waveguides with Small Discontinuities

Sergey S. Kurennoy
University of Maryland, Physics Department,
College Park, MD 20742, USA

Abstract

Trapped modes are studied in a waveguide with many small discontinuities, which is a good model for the vacuum chamber of large accelerators. Frequencies of trapped modes and their resonance contributions to the coupling impedance are calculated.

1 INTRODUCTION

It has been demonstrated recently [1] that a single small discontinuity (such as an enlargement or a hole) on a smooth waveguide results in the appearance of trapped electromagnetic modes with frequencies slightly below the waveguide cutoff frequencies, and that narrow resonances of the coupling impedance near the cutoff can be attributed to these trapped modes. This phenomenon could be dangerous for beam stability in large superconducting proton colliders like LHC, where the design anticipates a thermal screen (liner), with many small pumping holes, inside the beam pipe [2]. In such structures with many small discontinuities and high wall conductivity due to inner copper coating, the trapped modes can contribute significantly to the coupling impedances.

2 A SINGLE DISCONTINUITY

We list some results from [1]. In a cylindrical waveguide with perfectly conducting walls having a small axisymmetric enlargement at $z = 0$, with characteristic dimension much smaller than the pipe radius b, there is a solution of the Maxwell equations with frequency Ω_1 slightly below the cutoff frequency $\omega_1 = \mu_1 c/b$ (μ_m is the mth root of the Bessel function J_0). Far from the discontinuity (in fact, for $|z| > b$) the fields of the TM trapped mode are

$$\mathcal{E}_z^{(1)} = (\mu_1/b)^2 J_0(\mu_1 r/b)\exp(-k_1|z|) \ , \qquad (1)$$

and $\mathcal{E}_r^{(1)}, \mathcal{H}_\theta^{(1)}$ with corresponding radial behavior. The "propagation" constant $k_1 = \sqrt{\omega_1^2 - \Omega_1^2}/c$ is

$$k_1 = \mu_1^2 A/b^3 \ , \qquad (2)$$

where A is the area of the cross section of the enlargement in the rz-plane. Note that A enters Eq. (2) with its sign; e.g., for an iris that protrudes into the pipe, A would have a negative sign, and solution (1) would not exist. We assume from the beginning that $k_1 b \ll 1$. So, the trapped mode is spread along the axis of the pipe over the long distance $l_1 \equiv k_1^{-1} \gg b$. From (2) frequency shift $\Delta\omega_1 = \omega_1 - \Omega_1$ is

$$\Delta\omega_1 = \omega_1 \mu_1^2 (A/b^2)^2/2 \ . \qquad (3)$$

For the case of a finite, though large, conductivity of the walls, the trapped mode exists only if damping rate γ_1 is smaller than $\Delta\omega_1$, i.e. when $\gamma_1 = \omega_1\delta/(2b) < \Delta\omega_1$, where $\delta = \sqrt{2/(\mu_0\sigma\omega_1)}$ is the skin depth in the pipe wall.

This trapped mode produces a narrow resonance of the longitudinal coupling impedance with the peak value

$$R_1 = \frac{4 Z_0 \mu_1 A^3}{\pi\delta b^5 J_1^2(\mu_1)} \ . \qquad (4)$$

It was shown that a small hole in the pipe wall also creates localized axisymmetric trapped modes. Results for an enlargement remain valid for the hole if we substitute $A \to \alpha_\theta/(4\pi b)$, where α_θ is the magnetic susceptibility of the hole, in Eqs. (2)–(4). A similar study has been performed for higher-order and TE waveguide modes, and the existence of trapped modes was also demonstrated [1].

3 MANY DISCONTINUITIES

Consider an axisymmetric waveguide with N small enlargements located at z_i and having areas A_i of the longitudinal cross section, $i = 1, 2, \ldots, N$. In this structure, we look for a solution of the Maxwell equations with frequency Ω below the cutoff ω_1 in the piece-wise form (the radial behavior is given by Eq. (1)): $a_1 e^{kz}$ for $z < z_1$, $a_{n+1}e^{kz} + b_n e^{-kz}$ for $z_n < z < z_{n+1}$, and $b_N e^{-kz}$ for $z > z_N$, where $k = \sqrt{\omega_1^2 - \Omega^2}/c > 0$, and a_i, b_i, are amplitudes to be determined. We assume $kb \ll 1$ and enlargements are separated by distances larger than the chamber diameter, so that one can neglect higher modes.

To find the eigenfrequency of the trapped mode we use continuity conditions and the Lorentz reciprocity theorem, e.g. [3]. It gives $2N$ simultaneous homogeneous equations for $2N+1$ variables (a_i, b_i and k). The condition for the solutions for a_i, b_i to exist, i.e. the determinant of the matrix in the LHS to vanish, gives an equation for k, which can be written recurrently for any N. In notations $y_i = d_i/x$ with $d_i = \mu_1^2 A_i/b^2 \ll 1$ and $x = kb$, Eq. (2) for $N = 1$ becomes $1 - y = 0$. For $N = 2$

$$D_{1,2}(k) \equiv (1-y_1)(1-y_2) - e^{-2kg_{1,2}} y_1 y_2 = 0 \ , \qquad (5)$$

where $g_{i,k} = z_k - z_i$, ($k > i$), is a longitudinal distance between i-th and k-th discontinuities. Similarly, for $N = 3$

$$D_{1,3}(k) \equiv D_{1,2}(k) D_{2,3}(k) - e^{-2kg_{1,3}} y_1 y_3 = 0 \ . \qquad (6)$$

By induction, for $N > 3$ discontinuities

$$D_{1,N}(k) \equiv D_{1,N-1}(k) D_{N-1,N}(k) - \qquad (7)$$
$$\sum_{m=2}^{N-2} D_{1,m}(k) e^{-2kg_{m,N}} y_m y_N - e^{-2kg_{1,N}} y_1 y_N = 0 \ .$$

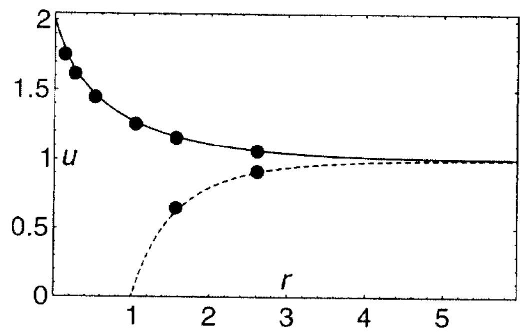

Figure 1: Ratio k/k_1 versus g/l_1 for symmetric and anti-symmetric (dashed) modes. Thick points show numerical results.

3.1 N=2

Let us introduce new variables: $\rho = A_2/A_1 > 1$; $d = \mu_1^2 A_1/b^2$; $u = x/d = k/k_1$, and $r = gd/b = g/l_1$, cf. Eq. (2). Then Eq. (5) takes the form

$$(u-1)(u-\rho) - \rho\exp(-2ur) = 0 \ . \qquad (8)$$

There are two positive solutions: u_s, which exists for any $r > 0$, and decreases asymptotically from $\rho + 1$ at small r to ρ when $r \gg 1/\rho$; and u_a, which exists only for $r > (\rho+1)/(2\rho)$, and increases from 0 to 1 with r increase. The asymptotic values ρ and 1 correspond to the two independent trapped modes for the two far separated discontinuities, see Eq. (2). For two identical discontinuities, $\rho = 1$, the factorized equation is

$$[u - 1 - \exp(-ur)]\,[u - 1 + \exp(-ur)] = 0 \ , \qquad (9)$$

and both its solution tends to 1 at large r, see Fig. 1. Solution u_s gives symmetric fields, and u_a antisymmetric ones, i.e. fields are zero in the midpoint between the two identical enlargements, Fig. 2. The behavior of u_s at small r is easy to explain: two close enlargements work like a single one with area $A = A_1 + A_2$. It corresponds to $u_s \to \rho + 1$, when $r \to 0$.

We have calculated numerically the lowest eigenfrequencies in a long cylindrical resonator with two small pill-boxes using the code SUPERFISH [5]. To exclude the influence of the side walls, one has to choose length L of the resonator to be large, $L \gg l_1 = b^3/(\mu_1^2 A)$. We used $b = 2$ cm, $A_1 = A_2 = 0.18$ cm^2, g =1-20 cm and L from 40 cm to 100 cm. Fig. 1 shows that numerical and analytical results are in good agreement.

The resonant contributions of trapped modes to the coupling impedance can be calculated as for a cavity with known eigenmodes, e.g. [4]:

$$R_2 = R_1 u^3 \frac{u(1+\rho) + 2\rho\,[\exp(-ur)\cos(\mu_1 g/b) - 1]}{u(1+\rho) + 2\rho\,[\exp(-2ur)(1+ur) - 1]} \ , \qquad (10)$$

where R_1 is the impedance for a single enlargement with area A_1, cf. Eq. (4), and $u = u(r,\rho)$ is a solution of Eq. (8). For small r the ratio R_2/R_1 tends to $(1+\rho)^3$ for the "symmetric" solution u_s. For large distances, R_2/R_1 becomes ρ^3 for u_s and 1 for u_a. There are some oscillations at intermediate distances. For two identical discontinuities, $\rho = 1$, ratio R_2/R_1 at large distances becomes $(1 \pm \cos(\mu_1 g/b))$. While the sum of the impedances is just twice the impedance of a single enlargement, there are strong oscillations for each of two modes.

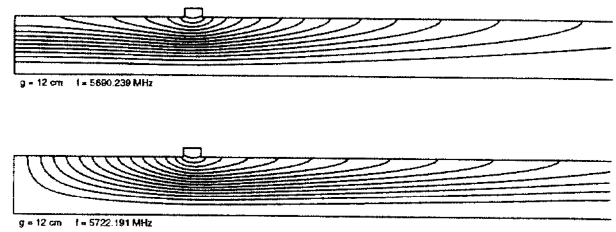

Figure 2: Electric field lines for symmetric (top) and anti-symmetric (bottom) trapped modes.

3.2 N=3

We condsider only the case of three identical equidistant discontinuities, i.e. $d_i = d$, $i = 1,2,3$ and $g_{1,2} = g_{2,3} = g$. Equation (6) transforms into

$$(1-y)(u-1+e^{-2ur})\left[(u-1)^2 - (u+1)e^{-2ur}\right] = 0 \ . \quad (11)$$

The second brackets give an antisymmetric mode for two enlargements separated by $2g$, cf. Eq. (9). The square brackets give two symmetric trapped modes: u_{s0} corresponds to fields without nodes, exists for all $r > 0$, and tends to 3 at small r; and u_{s1}, which exists only when $r > 3/2$ and has fields with 2 nodes. All three solution goes to 1 at large distances between discontinuities.

The impedance for the symmetric modes is

$$R_3 = R_1 u \frac{\left(e^{ur}(u-1) + e^{-ur} + 2u\cos(\mu_1 g/b)\right)^2}{3u + 1 - e^{-2ur} + 4ur(u-1)} \ . \qquad (12)$$

At small distances, R_3/R_1 goes to $3^3 = 27$ for u_{s0}. At large r it oscillates as $(1 \pm \sqrt{2}\cos(\mu_1 g/b))^2/2$ for u_{s0}, u_{s1}, see Fig. 3. In spite of the oscillations for each of the trapped modes, the sum of the impedances becomes just triple of that for a single discontinuity at large spacings in which case all three modes have the same frequency, Eq. (3).

3.3 Many Identical Discontinuities

In the case of N identical equidistant enlargements equation (7) can be factorized in the form

$$(1-y)^{N-2} P_n(y) P_m(y) = 0 \ , \qquad (13)$$

where $n = m = N/2$ for even N and $n = m + 1 = (N+1)/2$ for odd N, and $P_j(y)$ are polynomials in y of the power j except exponential dependence on $u = 1/y$ in their coefficients, cf. Eqs. (9) and (11). Equation $P_n(y) = 0$ has up to n positive solutions corresponding to symmetric trapped modes. The actual number of the roots depends

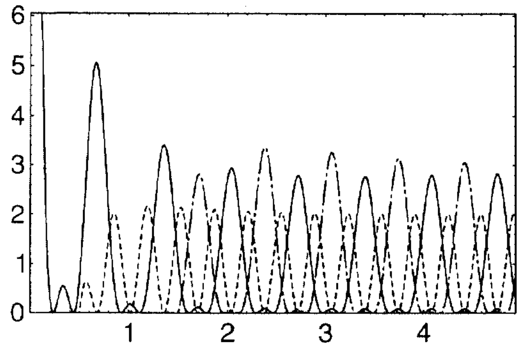

Figure 3: Ratio R_3/R_1 versus g/l_1: s_0 solid, a dashed, and s_1 dash-dotted line.

on the distance g between discontinuities. For any g there is at least one solution, and it behaves like $y \simeq 1/N$ at small distances, i.e. $k \simeq Nk_1$, because $P_n(y) \to 1 - Ny$ when $g/l_1 \to 0$. This solution corresponds to the maximal symmetric trapped mode, without nodes. It always has the largest frequency shift, i.e. the lowest frequency between all the trapped modes, and the impedance N^3 times that for a single discontinuity, Eq. (4), when $g/l_1 \to 0$.

Equation $P_m(y) = 0$ gives up to m solutions corresponding to antisymmetric trapped modes. At large distances, when $g/l_1 \gg 1$, the asymptotics of $P_j(y)$, $j = n, m$, are $(1-y)^j$, and there are $N = n + m$ solutions of Eq. (13) which asymptotically tend to 1.

4 PERIODIC STRUCTURES

4.1 One Discontinuity per Period

Consider now periodic arrays of discontinuities. We assume that the period of the structure D is longer than the waveguide diameter, $D > 2b$, and look for a periodic (with the same period D) solution of the Maxwell equations with frequency Ω below the waveguide cutoff, $\Omega < \omega_1$. Applying the reciprocity theorem and continuity conditions leads to a simple equation for k:

$$u = (1 + e^{-up})/(1 - e^{-up}) \ , \qquad (14)$$

where $u = 1/y = k/k_1$ and $p = dD/b = D/l_1$. This equation has only one positive solution $u = u(p) > 1$ for any positive value of p. It tends to 1 for $p \gg 1$, but its asymptotics at $p \ll 1$ is $u(p) \simeq \sqrt{2/p} = \sqrt{2l_1/D}$, that is quite different from those for a finite number N of discontinuities ($u \to N$, see Sect. 3). Since $u = k/k_1 = l_1/l$, where $l \equiv 1/k$, it has a meaning of the number of effectively interacting discontinuities. It also gives a new "effective" length of the trapped mode in a periodic structure: $l \simeq \sqrt{Dl_1/2} = \sqrt{Db^3/(2\mu_1^2 A)}$. The frequency shift down from the cutoff frequency for this trapped mode instead of Eq. (3) becomes

$$\Delta\omega = \omega_1 A/(bD) \ . \qquad (15)$$

We checked Eq. (15) by numerical computations treating one structure period as a closed resonator, because metallic end walls placed in midpoints between enlargements would not disturb the fields. The results agree well.

The resonant coupling impedance per period is a rather complicated expression, see [6]. Its asymptotics are: for short distances ($p \ll 1$),

$$R_p \to \frac{Z_0}{\pi\mu_1^2 J_1^2(\mu_1)} \frac{2b}{\delta} \frac{\sin^2[\mu_1 D/(2b)]}{\mu_1 D/(2b)} \ ,$$

that is independent of enlargement area A, except that this asymptotic is valid when $2b < D \ll l_1 = b^3/(\mu_1^2 A)$, while in the opposite extreme ($p \gg 1$), $R_p \to R_1 \propto A^3$. Since A is small ($d = \mu_1^2 A/b^2 \ll 1$), the impedance per period is much larger for short-period structures.

4.2 A Few Discontinuities per Period

In the case when there are N enlargements per period, a system of $2N$ homogeneous equations differs from that in Section 3 only by two first and two last equations, due to periodicity, and is studied in the same way. For example, for $N = 2$, the equation for k takes the form:

$$(1-y_1)(1-y_2) - e^{-2kg} y_1 y_2 - 2e^{-kD} \qquad (16)$$
$$+ e^{-2kD}(1+y_1)(1+y_2) - e^{-2k(D-g)} y_1 y_2 = 0 \ ,$$

where g is the distance between the discontinuities, $g \leq D$. When discontinuities are identical, $y_1 = y_2 = y$, it factorizes into two equations ($u = 1/y$):

$$u = \left[1 + e^{-up} \pm (e^{-ur} + e^{-u(p-r)})\right] / (1 - e^{-up}) \ , \quad (17)$$

where $p = dD/b$ and $r = dg/b$, $r \leq p$. The first of them always has a solution, corresponding to a symmetric mode. The second equation adds an antisymmetric one. We missed this mode in Section 4.1, because its period is twice longer than the structure period. The antisymmetric mode exists when (i) p is large enough, and (ii) both $\kappa = r/p = g/D$ and $(1-\kappa)$ are not very small.

5 CONCLUSIONS

Trapped modes in waveguides with many small discontinuities such as enlargements or holes are studied for periodic and aperiodic structures, see [6] for more details. Calculated eigenfrequencies are in good agreement with numerical computations. Most results work also for TE- and higher-order modes. The results are applied to obtain coupling impedance estimates for the liners (thermal screens) of large superconducting colliders at frequencies near the cutoff, see in Refs. [6, 7].

6 REFERENCES

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