Harmonic Analysis of Elongated Rectangular Aperture Magnets

A. Chirkov, N. Doinikov, Yu. Severgin, S. Sytchevsky
Efremov Scientific Research Institute of Electrophysical Apparatus, 189631 St.-Petersburg, Russia

Abstract

An analytical solution for the field with an antisymmetry plane in an infinite rectangular channel (2-D plane problem) has been found by Fourier method. The flux function distribution over the channel boundary specified in an analytical form or in a number of discrete points serves as the basic information for calculations. The harmonic field representation in the vicinity of the channel axis is given. The authenticity of the results obtained is illustrated using dipole,quadrupole and sextupole fields. The problems of numerical realization of the technique proposed are discussed. The developed software can be used for a more detailed analysis of numerical results and for the treatment of magnetic measurement data. It can be effectively used for the field analysis in rather elongated rectangular regions.

1 INTRODUCTION

The harmonic field representation is widely used in the optics of charged particle beams [1], with one summand usually predominating, whereas other summands form minor disturbances in respect of it. Thus, the dipole field is described by a linear term, the quadrupole- by the quadratic one etc. In real magnetic systems besides the main harmonics the whole spectrum, this directly being is present to this or that extent related to their design features and manufacturing precision. The field deviation from an ideal distribution is the most clear in the vicinity of disturbance sources. By taking the areas located near the poles and excitation coils into the boundaries of the region considered it is possible to obtain an effective analysis of field disturbances. The field analysis in a circular aperture has been widely spread [2]. But, in a number of cases there occur the channels of the rectangular and elliptic cross-section. The field analysis in these channels fails in accuracy in passing to the inscribed circular cross-section. The field expansion in the rectangular aperture has been obtained by its known distribution over the boundary. The solution for the systems with antisymmetry plane has been found. The harmonic field analysis has been performed. In order to verify the results obtained their adequacy to particular cases of the dipole, quadrupole and sextupole fields has been determined.

2 GENERAL CONSIDERATIONS.

The transverse field for the translation symmetry is described by the potential vector component $A_z = A(x,y)$, (x, y are the transverse and z is the longitudinal coordinates). The component in its meaning is a flux function. It is supposed that considered are the magnetic systems with the median plane: $y = 0$. The solution of quite a general boundary problem (see Fig.1) may be represented as a superposition of two solutions:

$$A = A_1 + A_2 \qquad (1)$$

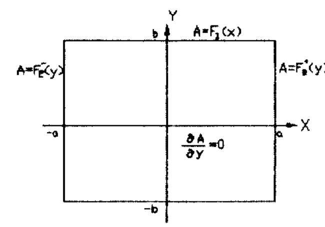

Figure 1:

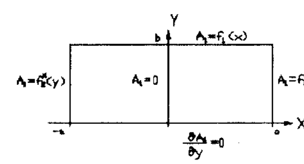

Figure 2:

The first solution, A_1 (see Fig.2), describes the antisymmetrical pole relative to the plane $x = 0$, the second solution, A_2 (see Fig.3), describes the symmetrical field. The above solutions should satisfy the following conditions:

$$A_1|_{x=0} = 0$$
$$\frac{\partial A_z}{\partial x}|_{x=0} = 0$$

which allow only the first quadrant of the coordinate plane to be considered. (Note, that A_1 is an odd and A_2 is an even function). It is necessary to form the boundary conditions. For A_1 there is:

$$A_1(x,b) = f_1^*(x) = [F_1(x) - F_1(-x)]/2$$
$$A_1(a,y) = f_2^*(y) = [F_2^+(y) - F_2^-(y)]/2 \qquad (2)$$

From the first equality of Eq.2 it is evident, that:

$$f_1^*(-x) = -f_1^*(x)$$

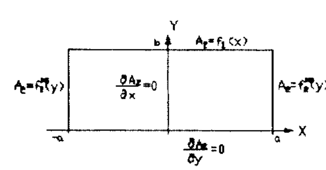

Figure 3:

It follows from the oddness condition of A_1 that:

$$A_1(-a,y) = -A_1(a,y)$$

and

$$A_1(0,y) = 0$$

For A_2 there is:

$$A_2(x,b) = f_1^{**}(x) = [F_1(x) + F_1(-x)]/2$$

$$A_2(a,y) = f_2^{**}(y) = [F_2^+(y) + F_2^-(y)]/2 \qquad (3)$$

It is evident from the first equality of Eq.3, that

$$f_1^{**}(-x) = f_1^{**}(x).$$

It follows from the evenness condition of A_2,that:

$$A_2(-a,y) = A_2(a,y)$$

and

$$\frac{\partial A_z}{\partial x}|_{x=0} = 0$$

3 SOLUTION RESULTS.

It is possible [3,4] to get the solution of $A_1(x,y)$ satisfying Laplace equation and boundary conditions:

The solution of A_2 can be obtained likewise. According to the conditions on the coordinate planes it should by of a symmetrical form relative to the variables x/a and y/b:

Using the known expansions of trigonometric and hyperbolic functions it is possible to represent A(x,0) as a series:

$$A(x,0) = \sum_{k=1}^{\infty} \frac{a_k}{k}\left[\frac{x}{a}\right]^k + a_0 \qquad (4),$$

where the summands with odd powers would be conditioned by the A_1 solution and those with even powers-by the A_2 solutions:

$$a_{2k+1} = \frac{\pi^{2k+1}}{(2k)!}\left\{(-1)^k \sum_{n=1}^{\infty} M_n \frac{n^{2k+1}}{ch\frac{n\pi b}{a}} + \right.$$

$$\left. + \left(\frac{a}{2b}\right)^{2k+1} \sum_{n=1}^{\infty} N_n \frac{(2n-1)^{2k+1}}{ch\frac{(2n-1)\pi a}{2b}}\right\} \qquad (5)$$

$$a_{2k} = \frac{(\pi/2)^{2k}}{(2k-1)!}\left\{(-1)^k \sum_{n=1}^{\infty} C_n \frac{(2n-1)^{2k}}{ch\frac{(2n-1)\pi b}{2a}} + \right.$$

$$\left. + \left(\frac{a}{b}\right)^{2k} \sum_{n=1}^{\infty} D_n \frac{2n-1)^{2k}}{ch\frac{(2n-1)\pi a}{2b}}\right\} \qquad (6)$$

(for $k = 0, 1, 2, 3, \ldots$) where:

$$\left.\begin{aligned} M_n &= \frac{2}{a}\int_0^a f_1^*(x) sin\frac{n\pi x}{a} dx \\ N_n &= \frac{2}{b}\int_0^b f_2^*(y) cos\frac{(2n-1)\pi y}{2b} dy \end{aligned}\right\}$$

$$\left.\begin{aligned} C_n &= \frac{2}{a}\int_0^a f_1^{**}(x) cos\frac{(2n-1)\pi x}{2a} dx \\ D_n &= \frac{2}{b}\int_0^b f_2^{**}(y) cos\frac{(2n-1)\pi y}{2b} dy \end{aligned}\right\}$$

4 SPECIAL FEATURES OF NUMERICAL REALIZATION.

Direct calculation of the coefficients a_{2k} and a_{2k+1} by Eqs.(5) and (6) gives rise to calculation errors accounted for the presentation of large numbers on a computer even at k=4. This effect is especially marked in case of elongated rectangles.

These formulas should be modified. The introduction of factors under the summation sign permits, for example, the following for the first series in a_{2k+1} to be obtained:

$$\sum_{n=1}^{\infty} M_n \frac{\pi^{2k+1}}{(2k)!}\frac{n^{2k+1}}{ch\frac{n\pi b}{a}} = M_n'$$

Stirling formula:

$$n! \approx n\sqrt[n]{2\pi n} exp(-n + (1/12)n^2 - (1/360)n^3 + \ldots)$$

yields

$$M_n' = \sum_{n=1}^{\infty} M_n \left(\frac{n\pi e}{2k}\right)^{2k+1} \sqrt{k/\pi}\frac{1}{exp(1+1/24k)ch\frac{n\pi b}{a}} \qquad (7)$$

Similar expressions may be derived also for other terms of the series a_{2k+1} and a_{2k}. The Stirling formula may be used at large values of n; at small values of n it yields an unacceptable large error (at $n = 4$ the error amounts to 10^{-5}). That is why, for $k = 1, 2, 3$ values the calculation are performed using Eqs.(5) and (6),and for $\kappa \geq 4$ the formula similar to Eq.(7) is used.

For elongated regions the series in Eqs.(7) have a bad converge.For the summation termination conditions of these series ($b_n/s_n \leq \epsilon$, where b_n is the series term, s_n is the partial s satisfied it is necessary to calculate the terms of the series with the numbers n=50 (a/b=1/2), n=100 (a/b=1/5), n=200 (a/b=1/8), this resulting in calculation errors. To find the sum of the above series the following approach has been suggested. By representing, for example, the multiplier at the coefficient M_n in Eq.(5) as:

$$b_n = \frac{Qn^{2k+1}}{ch\frac{n\pi b}{a}} \quad , \quad Q = \frac{\pi^{2k+1}}{(2k)!} \quad ,$$

and representing $ch\frac{n\pi b}{a} \approx 1/2 \cdot exp(\frac{n\pi b}{a})$ (at $n = 40$ and $a/b = 1/8$ an error amounts to $\approx 10^{-7}$) the following is obtained:

$$b_n = \frac{2Qn^{2k+1}}{exp(\frac{n\pi b}{a})} = \frac{2Q((n-1)+1)^{2k+1}}{exp(\frac{(n-1)\pi b}{a})exp(\frac{\pi b}{a})} =$$

$$\frac{2Q\sum_{m=0}^{2k+1}(n-1)^m \left|\frac{m}{2k+1}\right|}{exp(\frac{(n-1)\pi b}{a})exp(\frac{\pi b}{a})} = \frac{b_{n-1}}{exp(\frac{\pi b}{a})}\sum_{m=0}^{2k+1}\frac{\left|\frac{m}{2k+1}\right|}{(n-1)^{2k+1-m}} \qquad (8).$$

Thus, the representation of b_n is obtained by b_{n-1}. Consider the series in (8)

$$\sum_{m=0}^{k+1} a_m$$

$$a_m =$$

$$= \frac{(2k+1)!}{m!(2k+1-m)!(n-1)^{2k+1-m}} = a_{m+1}\frac{m+1}{(2k+1-m)(n-1)},$$

Knowing that $a_{2k+1} = 1$, it is easy to calculate a_m for all $m = 0, 1, 2, ..., 2k$.

And finally, we get for Eq.(8):

$$b_n = \frac{b_{(n-1)}}{exp(\frac{\pi b}{a})}\sum_{m=0}^{2k+1} a_m \quad , \quad a_m = a_{m+1}\frac{m+1}{(2k+1-m)(n-1)};$$

$$a_{2k+1} = 1$$

Thus, for k values exceeding a given level the calculations by Eqs.(7) are replaced with the calculations by Eqs.(9).

5 SOFTWARE REALIZATION.

An automated procedure for the calculation of the a_k coefficients is realized in the form of the program module set using a standard dialect of the FORTRAN algorithmic language. Access to the FOURIE subprogram allows all the required coefficients to be obtained immediately.

In the FOURIE subprogram the assignment of corresponding components of the magnetic induction vector is provided instead of the vector potential assignment .

6 RESULTS OF NUMERICAL EXPERIMENTS.

The FOURIE subprogram has been tested for the ideal dipole, quadrupole and sextupole fields.

a,b - dimension of the rectangle,on whose sides the vector potential distribution is given:

a) a=1. b=1.

serial	Expansion coefficients		
number	dipole	quadrupole	sextupole
0	$2 \cdot 10^{-10}$	$2.0 \cdot 10^{-10}$	$2.0 \cdot 10^{-10}$
1	1.00000009	$4.7124 \cdot 10^{-10}$	$1.3986 \cdot 10^{-10}$
2	0	1.00000001	0
3	$3.8146 \cdot 10^{-7}$	$-1.3565 \cdot 10^{-9}$	$-4.99999 \cdot 10^{-1}$
4	$2.0293 \cdot 10^{-10}$	$2.0293 \cdot 10^{-10}$	$2.0293 \cdot 10^{-10}$
5	$-2.1666 \cdot 10^{-7}$	$1.3149 \cdot 10^{-9}$	$-6.4992 \cdot 10^{-8}$
6	0	$-1.4172 \cdot 10^{-8}$	0
7	$-1.7969 \cdot 10^{-7}$	0	$-1.4705 \cdot 10^{-8}$
8	$2.0 \cdot 10^{-10}$	$-1.7160 \cdot 10^{-9}$	$2.0 \cdot 10^{-10}$
9	$5.2544 \cdot 10^{-8}$	$2.0 \cdot 10^{-10}$	$1.5297 \cdot 10^{-8}$
10	0	$3.9364 \cdot 10^{-9}$	0
11	$4.4456 \cdot 10^{-8}$	0	$1.3550 \cdot 10^{-8}$

b) a=5. b=1.

serial	Expansion coeficients		
number	dipole	quadrupole	sextupole
0	$2 \cdot 10^{-10}$	$2.6825 \cdot 10^{-8}$	$2.0 \cdot 10^{-10}$
1	5.00001	$1.0995 \cdot 10^{-9}$	$-2.5899 \cdot 10^{-7}$
2	$5.9217 \cdot 10^{-9}$	24.99991	$5.9217 \cdot 10^{-9}$
3	$-4.2656 \cdot 10^{-6}$	$2.2673 \cdot 10^{-8}$	-62.49985
4	$6.3519 \cdot 10^{-8}$	$1.5491 \cdot 10^{-5}$	$6.3518 \cdot 10^{-8}$
5	$2.0229 \cdot 10^{-5}$	$1.2579 \cdot 10^{-7}$	$-7.4351 \cdot 10^{-5}$
6	0	$-9.0331 \cdot 10^{-5}$	0
7	$-3.5958 \cdot 10^{-5}$	0	$1.2992 \cdot 10^{-4}$
8	$2.0 \cdot 10^{-10}$	$1.1941 \cdot 10^{-4}$	$2.0 \cdot 10^{-10}$
9	$4.5807 \cdot 10^{-5}$	$2.0 \cdot 10^{-10}$	$-1.7034 \cdot 10^{-4}$
10	0	$1.0299 \cdot 10^{-4}$	0
11	$2.5267 \cdot 10^{-5}$	0	$-9.0746 \cdot 10^{-5}$

c) a=8. b=1.

serial	Expansion coeficients		
number	dipole	quadrupole	sextupole
0	$2 \cdot 10^{-10}$	$7.8951 \cdot 10^{-10}$	$2.0 \cdot 10^{-10}$
1	8.00003	$1.5708 \cdot 10^{-9}$	$4.4008 \cdot 10^{-8}$
2	$1.5544 \cdot 10^{-8}$	63.9998	$1.5544 \cdot 10^{-8}$
3	$-1.9772 \cdot 10^{-7}$	$9.7670 \cdot 10^{-8}$	-255.9991
4	$4.1571 \cdot 10^{-7}$	$-3.9377 \cdot 10^{-6}$	$4.1571 \cdot 10^{-7}$
5	$1.0840 \cdot 10^{-3}$	$1.3070 \cdot 10^{-6}$	$-1.1005 \cdot 10^{-2}$
6	0	$2.3247 \cdot 10^{-5}$	0
7	$-1.7402 \cdot 10^{-5}$	0	$-1.9288 \cdot 10^{-4}$
8	$2.0 \cdot 10^{-10}$	$1.0950 \cdot 10^{-4}$	$2.0 \cdot 10^{-10}$
9	$4.0817 \cdot 10^{-5}$	$2.0 \cdot 10^{-10}$	$-4.2155 \cdot 10^{-4}$
10	0	$2.4587 \cdot 10^{-4}$	0
11	$9.9898 \cdot 10^{-5}$	0	$-9.9048 \cdot 10^{-4}$
12	$2.0 \cdot 10^{-10}$	$-8.1803 \cdot 10^{-4}$	$2.0 \cdot 10^{-10}$

d) The subprogramm has been tested also for the sum of the dipole, quadrupole and sextupole fields. With the refernce data of a=5. b=1. the following results have been obtained:

serial number	Expansion coeficients
0	$2.0175 \cdot 10^{-4}$
1	4.99935
2	24.99991
3	-62.50002
4	$-6.8301 \cdot 10^{-5}$
5	$7.6837 \cdot 10^{-4}$
6	$-3.4874 \cdot 10^{-4}$
7	$1.7852 \cdot 10^{-3}$
8	$-2.5999 \cdot 10^{-4}$
9	$1.7368 \cdot 10^{-3}$
10	$-2.2120 \cdot 10^{-4}$
11	$1.1581 \cdot 10^{-3}$
12	$-2.3167 \cdot 10^{-4}$

7 REFERENCES

[1] W. C. Elmore and M. W. Garrett, Rev. Sci. Instr., *1954, V. 25. p. 480.*

[2] J. Cobb, R. Cole, Spectroscopy of Quadrupole Magnets: In: The Int. Symp. on Magnet Techn., *(MT-1), 1965, Stanford, SLAG, pp.431-446.*

[3] G. A. Grinberg Izbrannye voprosy matematichecheskoi teorii elektricheskikh and magnitnykh yavleny (On the mathematical theory of electric and magnetic phenomena), *M.-L.:Published by AN USSR 1948.*

[4] L. E. Tsyrlin, Izbrannye zadachi rascheta elektricheskikh and magnitnykh poley. (Some selected problems of electric and magnetic field calculations.), *M.: Sovetskoe Radio 1977.*

[5] G. E. Forsythe, M. A. Malcolm, C. B. Moler, Computer method for mathematical computations Prentice-hall, *INC., Englewood cliffs,N.J.07632, 1977*