An Alternative Formula for the Synchrotron Radiation Integral[1]

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Abstract

The classical expression for the field radiated by an accelerated electron is usually calculated using simplifying approximations.

A new form is proposed here with no approximation. It can be used to calculate the field at short range, and to estimate the emittance effects.

1. INTRODUCTION

The electromagnetic field emitted by a relativistic particle, as described by the Lienard-Wiechert formula, has two terms.

$$\vec{E}(T)=\frac{e}{4\pi\varepsilon_0 c}\left[\frac{1}{r}\,\frac{\vec{n}\times\left[\left(\vec{n}-\vec{\beta}\right)\times\vec{\beta}'\right]}{\left(1-\vec{n}.\vec{\beta}\right)^3}+\frac{c}{\gamma^2 r^2}\,\frac{\vec{n}-\vec{\beta}}{\left(1-\vec{n}.\vec{\beta}\right)^3}\right] \quad (1)$$

The field emitted at time t is sensed at the target at T = t+r/c. The acceleration term, which accounts for the synchrotron radiation, decreases as the inverse of the distance. The Coulomb term, which only transports along with the particle the energy stored in the field, decreases as the inverse square of the distance.

The vector $r\,\vec{n}$ joining the particle to the target will be broken into two parts, a constant average, and a variation $\vec{\rho}$ along the source

$$r\ \vec{n} = r_o\ \vec{n}_o - \vec{\rho} \quad (2)$$

If the source is short compared with the distance to the target, $\vec{\rho}$ may be neglected, and $r\vec{n}$ is constant. The two terms of (1) are then easily separated. However for a long source, such as an undulator, the variation of r and $\vec{n}$ should be taken into account. This is done in section 2 for the power density of the radiation and for the total energy. The results are valid for small emission angles. In the same approximation, section 3 gives the time T as a function of t in the case of a sinusoidal field. The complete formula at infinity is given in section 4. Section 5 outlines the changes to be made for finite distance, and the emittance is considered in section 6.

2. RADIATED ENERGY

The power density through a surface element dA is the scalar product of the Poynting vector with the unit vector $\vec{N}$. The normal and tangential components, with respect to $\vec{N}$, of the field $\vec{E}$ and of the direction vector $\vec{n}$, which makes an angle θ with $\vec{N}$, will be clearly identified in the exact expression of the power density.

$$\frac{dP}{dA}=\frac{1}{\mu_o c}\left\{\cos\theta\left|\vec{E}_\perp-\frac{1}{2}\left(\vec{n}.\vec{E}\right).\,\vec{n}_\perp\right|^2 + \frac{1}{\cos\theta}\left[\vec{n}_\perp.\vec{E}_\perp-\frac{1}{2}\left(\vec{n}.\vec{E}\right)\sin^2\theta\right]^2 - \frac{1}{4}\,\frac{\sin^2\theta}{\cos\theta}\left(\vec{n}.\vec{E}\right)^2\right\} \quad (3)$$

The periodicity of the motion over one machine turn, allows one to expand $\vec{E}(T)$ in a Fourier series whose term of order k is

$$\vec{E}\ (k\omega_r)\ =\ \int_{-\pi/\omega_r}^{+\pi\omega_r}\vec{E}\ (T)\ e^{-ik\omega_s T}\ dT \quad (4)$$

It can be integrated using the differential expression of the field [1], modified here to take fully into account the variation of r and $\vec{n}$

$$\vec{E}\ (T)\ =\ \frac{e}{4\pi\varepsilon_o c}\left[\frac{d}{dT}\left(\frac{1}{r}\ \frac{\vec{n}-\vec{\beta}}{1-\vec{n}.\vec{\beta}}\right)+\frac{c}{r^2}\ \frac{\vec{n}}{1-\vec{n}.\vec{\beta}}\right] \quad (5)$$

It has been shown [2] that the second term of (5) may be neglected when $\lambda << r$, and the Fourier component $\vec{E}\ (\omega)$ is therefore

$$\vec{E}\ (\omega)\ \#\ \frac{e}{4\pi\varepsilon_o c}\left\{\left[\frac{1}{r}\ \frac{\vec{n}-\vec{\beta}}{1-\vec{n}.\vec{\beta}}\ .\ e^{-i\omega T}\right]_{-\pi/\omega_r}^{+\pi/\omega_r} + i\omega\int_{-\pi/\omega_r}^{+\pi/\omega_r}\frac{\vec{n}-\vec{\beta}}{r}\ e^{-i\omega T}\frac{dT}{1-\vec{n}.\vec{\beta}}\right\} \quad (6)$$

Since the initial formula (1) does not take into account the interactions of the field with the chamber walls, the validity of all that precedes is restricted to small values of θ, and we may then use it in the limiting case $\theta \approx 0$. The expression (3) can thus be simplified by putting $\theta = 0$. This leaves only the first bracket, where the relevant term is the derivative of (5). The total energy radiated over one turn is then given by

$$\frac{\partial^2 W}{\partial\omega\partial A}\ \#\ \frac{\alpha\hbar}{4\pi^2}\left|\int\left(\frac{\vec{n}-\vec{\beta}}{r}\right)_\perp\ e^{-i\omega T}\ dt\right|^2 \quad (7)$$

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where the integration limits have been left open, and the integrated term of (6) omitted, since it deals with end effects which will not be treated here. The equation (7) is close to the formula given in [1], with two slight differences :

a) The vector $\left(\vec{n}-\vec{\beta}\right)_{\perp}$ is used instead of $\vec{n}\times\left(\vec{n}\times\vec{\beta}\right)$, since the new reference is $\vec{N}$ instead of $\vec{n}$.

b) The variable distance r belongs to the integrand.

3. SINUSOIDAL UNDULATOR

In the case of an undulator with a sinusoidal field, the integration variable will be replaced by the phase δ of the field. The exponent in (7) can be approximated by

$$\omega T = \mu\delta - A \sin(2\delta) - B \cos\delta$$

$$\omega_o = 4\pi\, \gamma^2 \frac{c}{\lambda_u}$$

$$\mu = \left(\frac{\omega}{\omega_o}\right) . \left(1 + \frac{K^2}{2} + \gamma^2\, \theta^2\right) \tag{8}$$

$$A = \left(\frac{\omega}{\omega_o}\right) . \frac{K^2}{4}$$

$$B = \left(\frac{\omega}{\omega_o}\right) . 2K\, \gamma\, \theta \cos\varphi$$

where again terms of the order of θ^2 have been dropped and a constant phase substracted. The angles θ_o and φ_o define the orientation of $\vec{n}_o$ with respect to the axis of the undulator. μ represents the average slip of the particle with respect to the emitted photons, A its modulation and B represents the transverse motion.

4. SIMPLIFIED INTEGRATION

To begin with, I shall suppose that $\vec{N}$ lies along the axis of the undulator, and shall not consider the variation of r. However, to be prepared to do so, the integration will be performed piecewise, as in [3]. Using the symmetries of the harmonic functions, (7) is integrated to give

$$\frac{\partial^2 W}{\partial\omega\partial A} = \alpha\hbar\left(\frac{4}{\pi}\, \gamma\, \frac{\omega}{\omega_o}\, N\right)^2 . \left[\frac{1}{Nr}\, \frac{\sin(N\mu\pi)}{\sin(\mu\pi)}\right]^2 \left|\vec{F}(\mu,A,B)\right|^2$$

$$\vec{F} = \sin\left(\mu\frac{\pi}{2}\right) \left\{\begin{bmatrix} K \\ 0 \end{bmatrix} S_3 - \begin{bmatrix} \gamma\theta \cos\varphi \\ \gamma\theta \sin\varphi \end{bmatrix} - S_1\right\}$$
$$- i\, \cos\left(\mu\frac{\pi}{2}\right) \left\{\begin{bmatrix} K \\ 0 \end{bmatrix} S_4 - \begin{bmatrix} \gamma\theta \cos\varphi \\ \gamma\theta \sin\varphi \end{bmatrix} S_2\right\} \tag{9}$$

$$S_1 = \int_o^{\pi/2} \sin(\mu\delta + A \sin 2\delta) \sin(B \sin\delta)\, d\delta$$

$$S_2 = \int_o^{\pi/2} \cos(\mu\delta + A \sin 2\delta) \cos(B \sin\delta)\, d\delta$$

$$S_3 = \int_o^{\pi/2} \cos(\mu\delta + A \sin 2\delta) \cos(B \sin\delta) \cos\delta\, d\delta$$

$$S_4 = \int_o^{\pi/2} \sin(\mu\delta + A \sin 2\delta) \sin(B \sin\delta) \cos\delta\, d\delta$$

5. VARIABLE R

The square bracket of (9) comes from the sum of the factors exp(μ .2nπ) due to the non periodic part of the exponent, where r does not appear. The variation of r along the undulator will be taken into account by using a mean value r_n valid for each period, which will alter the sum.

$$\frac{\vec{n}-\vec{\beta}}{r} = \frac{r_o\, \vec{n}_o}{r^2} - \frac{\vec{\rho}}{r^2} - \frac{\vec{\beta}}{r} \tag{10}$$

and the bracket can no longer be separated from the vector $\vec{F}$. The first term introduces a $1/r_n^2$ contribution which modifies the line width. The last term modifies also the line width, but due to a $1/r_n$ contribution. The middle term is due to the transverse oscillation of the particle. Its contribution is expressed by two new integrals analog to S_3 and S_4, but where $\cos\delta$ is replaced by $\sin\delta$.

6. CONCLUSION

A minor modification of the usual formulae allows one to take into account the short range radiation. The integrals S_1 through S_4 can be calculated directly, or expanded in a series of Bessel functions.

The finite emittance of the beam should be taken into account by a modification of the origin of $r_o\, \vec{n}_o$ and the orientation of $\vec{\rho}$. It plays also on the exponent (7) by altering the angle θ_o.

7. REFERENCES

[1] J. Jackson, Classical electrodynamics, New York : Wiley, 1975, ch. 14.

[2] C. Wang, Y. Xiao, "On algorithms for undulator radiation calculation", in Proceedings of the International Conference on Synchrotron Radiation Sources, Indore, India, February 1992, p. 270.

[3] A.N. Didenko et al., "Radiation from relativistic electrons in a magnetic wiggler", Soviet Physics J.E.T.P., vol. 49, p. 973, 1979.