Self-Stabilization of Higher-Multipole Modes of Coherent Transverse Oscillations due to Nonlinear Coulomb Tune-Shift

E. Mustafin
Institute for High Energy Physics
Protvino, Moscow region, 142284, Russia

Abstract

Stability of coherent betatron oscillations of an intense coasting beam in a synchrotron is studied. Local space-charge field is treated by the Green's function method. Analysis is made of the dipole mode, as well as of any higher-multipole mode of coherent oscillations in the transverse phase-plane. Having the same source, coherent and incoherent space-charge effects are studied concurrently, i.e. the problem is approached in a self-consistent way. Generalized threshold map technique shows that given the absence of nonlinearity of external guide field: (1) it is the dipole mode exclusively which may turn unstable, (2) coherent betatron tune shift exactly compensates the incoherent one, and (3) the higher-multipole modes always occur well below Landau damping threshold, the latter being introduced by the incoherent nonlinear tune shift itself.

1 INTRODUCTION

It is a common matter to study interaction between a charged intense beam and resonant surrounding in terms of transverse impedance. But such an approach is sometimes inconvenient in studies of interaction between the beam and its local space-charge field. Method followed below is based on further development of approach of Ref.[1] and on a generalized threshold map technique applied in Ref.[2]. Though a 1D (flat) relativistic coasting beam is considered here, the method involved can be readily extended to a 2D (round) coasting beam as well. The latter results in a mere complication of analytical expressions and requests much more powerful calculation tools to get numerical solution. In order to make the approach more transparent, nonlinearities of external focusing fields are ignored here.

2 POTENTIAL OF A CHARGED PLANE PLATE

Potential of a flat relativistic beam can be derived from

$$\frac{d^2\Psi}{dz^2} = -\frac{4\pi}{\gamma^2 Ll}\rho(z). \qquad (1)$$

Here L is orbit length, l is beam "width", γ is relativistic factor, z is vertical coordinate, $\rho(z)$ is charge density function normalized by

$$\int_{-\infty}^{\infty}\rho(z)dz = eN,$$

e is charge of particle, N is number of particles in the beam. The Green's function $G(z,z')$ of Eq.1 is a potential of a charged plane sheet with a unit density placed at a coordinate z':

$$G(z-z') = \mid z-z' \mid /2.$$

Thus,

$$\Psi(z) = -\frac{2\pi}{\gamma^2 Ll}\int_{-\infty}^{\infty} \mid z-z' \mid \rho(z')dz'. \qquad (2)$$

3 INCOHERENT COULOMB TUNE SHIFT

Let us consider a stationary phase trajectory

$$z = A\cos\varphi, \qquad p = -m\gamma A\dot{\varphi}\sin\varphi.$$

A steady-state distribution function $F(A)$ depends only on the amplitude $A = \sqrt{z^2 + p^2/m^2\gamma^2\dot{\varphi}^2}$ and is normalized by

$$\int_0^{\infty} F(A)AdA = 1.$$

This steady-state distribution of charge, according to our assumption, imposes a Lorentz force:

$$eE(z) = -ed\Psi/dz.$$

From Eq.2 one gets

$$E(z) = \frac{eN}{\gamma^2 Ll}\int_{-\pi}^{\pi} d\varphi' \int_0^{\infty} A'dA'\frac{dF}{dA'}\cos\varphi' \mid z - A'\cos\varphi' \mid .$$

Now, incoherent Coulomb tune shift can be derived proceeding from equation of a stationary betatron oscillations:

$$\frac{d^2z}{dt^2} + Q_0^2\dot{\theta}^2 z = \frac{eE(z)}{m\gamma}.$$

By applying a standard averaging procedure one gets

$$\Delta Q_{inc}(A) = -\frac{e^2N}{m\gamma^3 Q_0\dot{\theta}}\frac{1}{A}\int_0^{\infty} K_1(A,A')\frac{dF}{dA'}A'dA'. \quad (3)$$

The kernel $K_n(A,A')$ is defined as

$$K_n(A,A') = \int_0^{\pi} d\varphi \int_0^{\pi} d\varphi' \mid A\cos\varphi - A'\cos\varphi' \mid \cos(n\varphi)\cos(n\varphi').$$

Eq.3 describes the incoherent tune shift due to space charge of a steady-state distribution of particles.

4 INTEGRAL EQUATION FOR MULTIPOLES

One can expand the Laplace image of a small perturbation of $F(A)$ into Fourier series

$$f(A,\varphi)=\sum f_n(A)\exp(-in\varphi).$$

Then, the linearized Vlasov equation results in

$$(\omega-k\dot{\theta}+n\dot{\varphi})f_n(A)=n\frac{e}{m\gamma\dot{\varphi}}\frac{1}{A}\frac{dF}{dA}\frac{1}{2\pi}\int_{-\pi}^{\pi}\Psi\exp(in\varphi)d\varphi.$$

In fact, potential Ψ is imposed by the entire perturbation $f(A,\varphi)$, but we consider only its n-th multipole component $f_n(A)\exp(-in\varphi)$. It is supposed to bring the dominant contribution to the potential. Using the Green's function one obtains

$$\begin{aligned}&f_n(A)\left[\tfrac{\omega}{\dot{\theta}}-k+nQ_0\right.\\&\left.-n\frac{e^2N}{m\gamma^3Ll}\frac{1}{\pi Q_0\dot{\theta}^2}\frac{1}{A}\int_0^\infty K_1(A,A')A'dA'\frac{dF}{dA'}A'dA'\right]\\&=-n\frac{e^2N}{m\gamma^3Ll}\frac{1}{\pi Q_0\dot{\theta}^2}\frac{1}{A}\frac{dF}{dA}\int_0^\infty K_n(A,A')f_n(A')A'dA'.\end{aligned}\tag{4}$$

It is easy to solve one-dimensional linear integral equation Eq.4 numerically. However, the case $n=1$ (the dipole mode) can be treated analytically:

$$\frac{\omega}{\dot{\theta}}=k-Q_0,\qquad f_1(A)\propto\frac{dF(A)}{dA}.$$

If we reconstruct the function of beam full-charge density ϱ which corresponds to the net distribution function $\mathcal{F}=F+f$,

$$\varrho(z)=\int dp\left[F(A)+f_1(A)\exp(-i\varphi)\right],$$

then $\varrho(z)=\rho(z+\delta z)$ where ρ is the charge density function corresponding to steady-state $F(A)$. This is the well-known solution for a dipole mode. It has a simple physical interpretation. Coherent dipole mode of oscillations is nothing but the beam center-of-mass oscillations at the unperturbed frequency Q_0, the latter being defined by the external magnetic lattice. In other words, the coherent frequency shift exactly compensates the incoherent one:

$$\Delta Q_{coh}=-\Delta Q_{inc}.\tag{5}$$

Note that this solution describes oscillations with a sustained amplitude. Other factors which were not included into consideration can well result in a growth of the oscillation amplitude.

5 GENERALIZED THRESHOLD MAP TECHNIQUE

In order to define whether the solutions of Eq.4 are stable or not, we use a generalized threshold map technique (GTMT). Let us introduce a formal parameter λ and function $y(A)$:

$$y(A)=\int_0^\infty K_n(A,A')f_n(A')A'dA'.$$

Then, Eq.4 can be put down as:

$$\lambda y(A)=\int_0^\infty\frac{K_n(A,A')\frac{dF}{dA'}y(A')dA'}{\Omega+\Omega_0(A')},\tag{6}$$

$$\Omega=-\frac{\frac{\omega}{\dot{\theta}}-k+nQ_0}{n\frac{e^2N}{m\gamma^3Ll}\frac{1}{\pi Q_0\dot{\theta}^2}},$$

$$\Omega_0(A)=\frac{1}{A}\int_0^\infty K_1(A,A')\frac{dF}{dA'}A'dA'.$$

Solution of Eq.6 is a discrete series of eigenvalues λ_s and corresponding eigenfunctions $y_s(A)$ for each value of Ω. We define function $\lambda(\Omega)$ as λ_s with the maximum absolute value for each definite Ω. Function $\lambda(\Omega)$ defines a curve in complex plane $\{\mathrm{Re}(\lambda),\mathrm{Im}(\lambda)\}$, Ω running from $-\infty$ to $+\infty$. According to GTMT, this curve is referred to as a threshold curve which splits complex plane $\{\mathrm{Re}(\lambda),\mathrm{Im}(\lambda)\}$ into unstable and stable regions. The first ones are the images of half-plane $\mathrm{Im}(\omega)>0$ subjected to mapping $\lambda(\Omega)$ and correspond to unstable solutions with growing amplitudes. The latter regions either correspond to damped solutions, or there are no resonance solutions for these λ at all. We look for solutions of Eq.6 with $\lambda=1$. Thus, GTMT gives a certain receipt: the solutions of Eq.6 would have stable or unstable character depending on the particular location of point $\lambda=1$ w.r.t. stable or unstable regions in $\{\mathrm{Re}(\lambda),\mathrm{Im}(\lambda)\}$.

6 NUMERICAL SOLUTION FOR VARIOUS $F(A)$

To solve Eq.6 numerically, one should take definite function $F(A)$ close to the realistic ones. Let us choose

$$F(A)=\frac{12}{1+a_1+a_1^2}\begin{cases}1-a^2/a_1, & 0\le a\le a_1;\\(1-a)^2/(1-a_1), & a_1\le a\le 1;\end{cases}$$

where $a=A/A_0$, A_0 is the maximum amplitude (the "half-height" of the beam), $a_1=A_1/A_0$. Such a choice provides various shapes of the steady-state distribution function $F(A)$ by varying free parameter A_1.

Fig.1 shows the threshold curves $\lambda(\Omega)$ for a_1 ranging from 0.1 (the inner curve) to 0.9 (the outer curve) in complex plane $\{\mathrm{Re}(\lambda),\mathrm{Im}(\lambda)\}$ for dipole mode $n=1$. Point $\lambda=1$ lays exactly on the threshold curve. As discussed above, this means that all the solutions would be oscillations of the beam center-of-mass with a sustained amplitude. However, any small displacement of this working point would result in solution becoming unstable. Thus, the solution of Eq.6 for dipole mode $n=1$ has, practically, an unstable character for all shapes of function $F(A)$ considered.

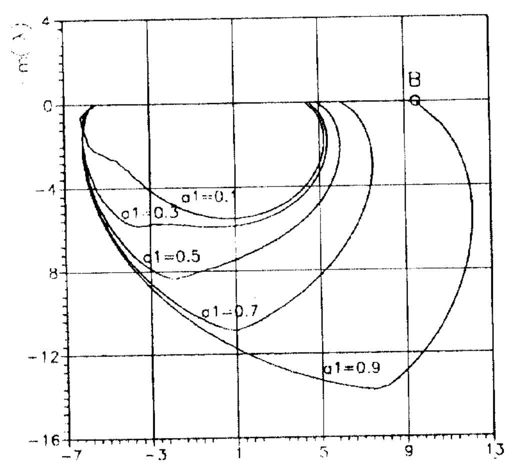

Figure 1: Threshold curves for $n = 1, a_1 = 0.1–0.9$.

Fig.2 shows similar threshold curves for quadrupole mode $n = 2$. Point $\lambda = 1$ is exactly on the threshold curves

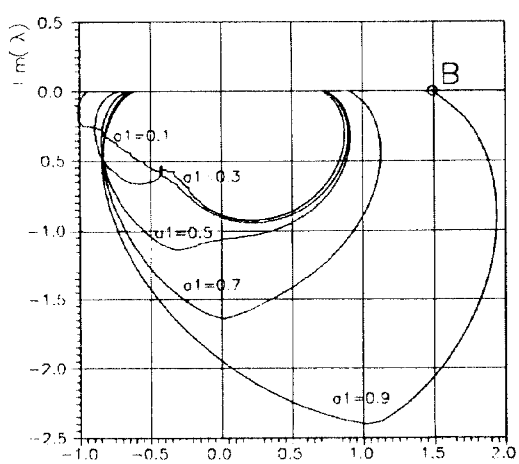

Figure 2: Threshold curves for $n = 2, a_1 = 0.1–0.9$.

again for values of $a_1 = 0.6–0.9$. This means that the beam with the "short-tail" distribution is unstable w.r.t. resonant excitation of the coherent quadrupole mode (oscillation of the beam shape). But for values of $a_1 = 0.1–0.5$ the point $\lambda = 1$ lays in the stable region, i.e. the beam with "long-tail" distribution is stable w.r.t. resonant excitation of the coherent mode $n = 2$. Let the point in which the threshold curves comes off the axis Re(λ) in Fig.2 be referred to as the "breaking-away" point B. Fig.3 shows B versus a_1 for modes $n = 1, 2, 3$. Curve B(a_1) for $n = 2$ starts from value B(0) < 1, then crosses B=1 at $a_1 = 0.6$ (cross-point C_2), and goes to infinity when $a_1 \to 1$. It is clear hereof that all distributions with $a_1 > C_2$ are unstable, while those with $a_1 < C_2$ are stable. The behavior of function B(a_1) for all modes $n = 1, 2, 3, \ldots$ has the same character. There is no cross-point C_1 for mode $n = 1$ because curve B(a_1) for $n = 1$ runs higher than line B=1. Cross-points C_i for modes $n \geq 3$ are always located to the right of point C_2 with $a_1 = 0.6$.

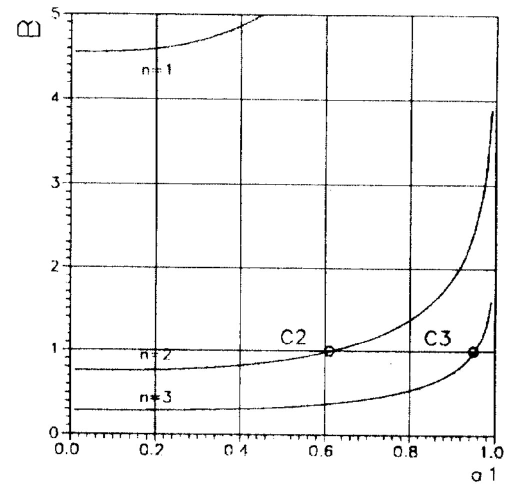

Figure 3: "Breaking-away" point B versus a_1 for n=1,2,3.

7 CONCLUSION

A (flat) beam is always unstable w.r.t. resonant excitation by Coulomb self-field of coherent dipole mode $n = 1$. However, if we assume that the beam has a long enough "tail" of distribution $F(A)$ (i.e. its parameter $a_1 < 0.6$) then all the higher coherent multipole modes, $n \geq 2$, would never be resonantly excited because of the Landau damping due to nonlinearities caused by the Coulomb self-field itself.

8 REFERENCES

[1] Balbekov V.I. *Effect of Positive Ions on the Beam Transverse Stability in Proton Synchrotrons.* Preprint IHEP 74-1, Serpukhov, 1974.

[2] Balbekov V.I., Ivanov S.V. *Thresholds of Longitudinal Instability of Bunched Beam in the Presence of Dominant Inductive Impedance.* Preprint IHEP 91-14, Protvino, 1991.