EFFECT OF SPACE CHARGE TUNE SHIFT ON TRANSVERSE INSTABILITIES *

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Abstract

Transverse instabilities in synchrotrons with large space charge tune shift are considered. In particular, the variation of the incoherent tune with longitudinal position in the bunch is included. Expansion in an appropriate basis set results in an eigenvalue problem which reduces to previously derived expressions when the space charge tune spread is ignored. The regime where the synchrotron tune is negligible compared to the space charge tune shift is also considered. Under appropriate conditions this results in a significantly smaller growth rate than the ones obtained using the weak coupling formalism. Finally, the effect of octupole induced betatron tune spread is included, resulting in a technique for estimating the maximum stable current.

1 MODERATE TUNE SHIFT REGIME

Consider the Vlasov equation for a ring containing M equally spaced, equally populated bunches. Betatron frequency ($\omega_\perp$) spread due to chromaticity (ξ) is included, but amplitude dependent tune spread is ignored. The synchrotron frequency (ω_s) is assumed to be independent of amplitude. Calculations are done in the smooth machine approximation so we neglect synchrobetatron resonances.

Since it follows individual particles, the Vlasov equation contains the incoherent betatron frequency

$$\omega_\perp(\delta,\phi) = \omega_y\left[1+\delta(\xi-\eta)\right] - \Delta\omega_I\rho(\phi)/\rho(0), \qquad (1)$$

where ω_y is the zero current betatron frequency for an on momentum particle, η is the frequency slip factor, $\Delta\omega_I$ is the incoherent space charge frequency depression in the center of the bunch, δ is the fractional momentum deviation, and $\phi = \theta - \omega_0 t$, where θ is azimuth, ω_0 is the revolution frequency of an on momentum particle, and $\rho(\phi)$ is the line density.

Using first order perturbation theory, and assuming a time dependence $\exp(-i\Omega t)$ yields an eigenvalue problem in four variables. Coupled bunch modes are identified, and the problem is reduced to studying the behavior of a single bunch. Using the dipole approximation for the transverse distribution and choosing the upper betatron sideband leaves an equation for $g_1(\phi,\delta)$; the longitudinal distribution function of the perturbation. The unperturbed longitudinal distribution is taken to be a Gaussian of rms width σ.

*Work supported by U.S. D.O.E.

We make the substitution $g_1 = g_2\exp[iQ_y(\xi/\eta-1)\phi]$ and express the result in normalized, Cartesian variables $z=\phi/\sqrt{2}\sigma$, $v=\dot{\phi}/\sqrt{2}\sigma\omega_s$,

$$\begin{aligned} &Qg_2(z,v) + \frac{\Delta\omega_I}{\omega_s}g_2 e^{-z^2} \\ &-\left(\frac{\Delta\omega_I}{\omega_s} - \frac{\Delta\omega_C}{\omega_s}\right)\frac{e^{-(z^2+v^2)}}{\sqrt{\pi}}\int_{-\infty}^{\infty} dv_1 g_2(z,v_1) \\ &+i\left\{v\frac{dg_2}{dz} - z\frac{dg_2}{dv}\right\} \\ &= -i\frac{e^{-(z^2+v^2)}}{\pi}\sum_k \hat{\chi}_k e^{i\tilde{k}\sqrt{2}\sigma z}D_k \end{aligned} \qquad (2)$$

where s is the coupled bunch mode number, $\tilde{k} = kM+s+Q_y(1-\xi/\eta)$, and $Q=(\Omega-\omega_y)/\omega_s$ is the frequency shift in units of the synchrotron frequency. Additionally,

$$\hat{\chi}_n = \frac{cq\bar{I}\hat{Z}_\perp[(nM+s)\omega_0+\omega_y]}{4\pi E_T\omega_s Q_y},$$

and

$$D_k = \int dz_1 dv_1 e^{-i\tilde{k}\sqrt{2}\sigma z_1}g_2(z_1,v_1),$$

where $\bar{I}$ is the average beam current, $\hat{Z}_\perp$ is the non-space charge part of the transverse impedance and $\Delta\omega_C$ is the coherent space charge frequency depression in the center of the bunch. The charge and total energy of a synchronous particle are q and E_T, respectively. The second term on the first line of equation (2) is due to the incoherent space charge tune spread and was absent in previous analyses. To solve the Vlasov equation expand g_2 as

$$g_2(z,v) = \sum_{n,m}(-i\sqrt{2})^n a_{n,m}H_n(z)H_m(v)e^{-(z^2+v^2)}, \qquad (3)$$

where

$$H_m(x) = e^{x^2}\left(\frac{d}{dx}\right)^m e^{-x^2},$$

is the Hermite polynomial of order m and the sum is over all pairs of non-negative integers. Substituting the expansion into the Vlasov equation, multiplying by $H_p(z)H_q(v)dzdv$ and integrating yields a matrix for the expansion coefficients [5]

$$-Qa_{p,q} = \sum_{n,m}T_{p,q,n,m}a_{n,m}.$$

The matrix element is given by

$$T_{p,q,n,m} = -\left\{\frac{n}{\sqrt{2}}\delta^q_{m+1}\delta^p_{n-1} + \sqrt{2}m\delta^q_{m-1}\delta^p_{n+1}\right\}$$
$$+i\frac{\delta^0_m\delta^0_q}{p!}\sum_k \hat{\chi}_k(\sigma\tilde{k})^{n+p}e^{-\tilde{k}^2\sigma^2}$$
$$+\frac{\delta^0_q\delta^0_m(\Delta\omega_C - \Delta\omega_I)}{\omega_s p!\sqrt{2\pi}}\Gamma\left(\frac{n+p+1}{2}\right)\mathrm{EV}(n+p)$$
$$+\frac{\delta^q_m\Delta\omega_I}{\omega_s p!\sqrt{2\pi}}\Gamma\left(\frac{n+p+1}{2}\right)\mathrm{EV}(n+p), \quad (4)$$

where δ^m_n is the Kronecker delta, $\Gamma(x) = (x-1)!$, and $\mathrm{EV}(k)$ is 1 when k is even and zero otherwise. The effect of space charge tune spread is contained in the last term on the r.h.s. of equation (4).

In practical applications the double sum in the eigenvalue problem needs to be truncated. In the zero current limit only the first term in the matrix T remains and the eigenvectors lie in subspaces of constant $p+q$. To allow for coupling between these synchrotron modes, without modifying the exact low intensity limit, the eigenvalue problem is truncated using

$$-Qa_{p,q} = \sum_{n+m\leq K} T_{p,q,n,m}a_{n,m}. \quad (5)$$

The value of K in equation (5) is equal to the largest synchrotron mode included in the calculation ie. the largest eigenvalue in the zero current limit. To solve equation (5) we define an array $N(p,q)$ which is one to one with the ordered pairs (p,q) for $p+q\leq K$. Using the index N results in the usual sort of matrix eigenvalue problem which we solve on the computer using standard techniques. We go on to establish the connection between the Hermite expansion technique and previous work.

Studies of beam stability usually employ action angle variables or, topologically equivalent, amplitude angle variables. Consider the amplitude angle variables r, ψ defined via $z = r\sin\psi$, $v = r\cos\psi$. Equation (2) is usually written

$$Qg_2(r,\psi) + i\frac{\partial g_2}{\partial \psi}$$
$$= -i\frac{e^{-r^2}}{\pi}\sum_k \chi_k e^{i\sqrt{2}\tilde{k}\sigma r\sin\psi}D_k, \quad (6)$$

where

$$D_k = \int r dr d\psi e^{-i\sqrt{2}\tilde{k}\sigma r\sin\psi}g_2(r,\psi),$$

and χ_k now includes the space charge impedance which is given by

$$Z_{\perp,SC} = i\frac{RZ_0}{\beta^2\gamma^2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right). \quad (7)$$

In equation (7) R is the radius of the synchrotron, a is the radius of a uniform equivalent beam, and b is the radius of the pipe. Space charge induced, betatron tune spread is not included in equation (6). Another, usually irrelevant, discrepancy between equations (2) and (6) is due to the fact that the Gaussian distribution does not have bounded support. The space charge force calculated via the summation is not identical to the second line of equation (2), since the tails of the adjoining bunches overlap the bunch under study. In what follows we assume that the equations following from (6) are appropriately modified to remove this discrepancy.

Instead of expanding in Hermite polynomials equation (3) would be given by

$$g_2(r,\psi) = \sum_{m,j}\tilde{a}_{m,j}r^{|m|}L^{|m|}_j(r^2)e^{im\psi - r^2}, \quad (8)$$

where $L^{|m|}_j$ is a generalized Laguerre polynomial. An infinite matrix for the expansion coefficients ($\tilde{T}$) can be obtained by substituting equation (8) into equation (6), multiplying by $rdrd\psi\exp(-in\psi)r^{|n|}L^{|n|}_k(r^2)$, and integrating [2]. Unlike the Hermite expansion technique, there is some freedom in defining the truncation procedure for the infinite matrix. A natural truncation is given by

$$-Q\tilde{a}_{n,j} = \sum_{|n|+2k\leq K}\tilde{T}_{m,j,n,k}\tilde{a}_{n,k}, \quad (9)$$

which includes all polynomials in r of order $\leq K$.

Since $r^{|m|}\exp(im\psi) = (v + i\,\mathrm{sgn}(m)z)^{|m|}$, equations (5) and (9) expand g_2 in equivalent basis sets, and, since $rdrd\psi = dzdv$, the measures used in the integrations to obtain the matrix elements are equivalent. It follows that (when the last term on the r.h.s. of equation (4) is neglected) $\tilde{a}$ and a are related by a linear transformation and that the eigenvalues of equations (5) and (9) are identical. We have verified this numerically. It can also be shown that equation (9) is equivalent to the transverse strong coupling-short bunch case considered by Wang [4].

Computer code has been written to solve equation (5) for $K \leq 28$ and studies of existing machines are in progress. One general feature of the studies is that the values of $\Delta\omega_I/\omega_s$ can be very large, especially at low energy. As a rule of thumb, we assume that the calculations become meaningful when $K \gtrsim \Delta\omega_I/\omega_s$, making the study of the large tune shift regime very difficult. In the next section we use an analytic approximation in this regime.

2 LARGE TUNE SHIFT REGIME

In the large tune shift regime $\Delta\omega_I \gtrsim \Delta\omega_C \gg \omega_s$. Instead of including a large number of synchrotron modes we set $\omega_s = 0$ and consider a beam which is frozen longitudinally. The vertical equation of motion is given by

$$\ddot{y} + \omega_y^2 y = \frac{F_\perp(x,y,\phi,t)}{\gamma m}, \quad (10)$$

where both transverse coordinates appear in the collective force. This force contains contributions due to currents flowing in the walls of the vacuum chamber and direct particle-particle forces which are responsible for the space charge tune spread. Averaging equation (10) over particles at a fixed value of ϕ causes the particle-particle contribution to cancel and results in an eigenvalue problem for $< y(\phi,t) >= y(\phi)\exp(-i\Omega t)$. Approximating $\Omega^2 - \omega_y^2 \approx 2\omega_y(\Omega - \omega_y)$ yields an eigenvalue problem for the upper betatron sideband

$$(\Omega - \omega_y)y(\phi) = -\frac{F_{wall}(\phi)}{2\omega_y \gamma m}. \tag{11}$$

Only the forces due to the currents in the wall remain and the coherent frequency is independent of $\Delta\omega_I$. Multiplying equation (2) by ω_s, letting $\omega_s \to 0$, and integrating over v results in an equivalent equation for the Gaussian line density.

For a cosine squared line density where the wall force is due to space charge in conjunction with a narrow band resonator, equation (11) may be solved exactly [5]. The coherent frequency with non-vanishing imaginary part is given by

$$\Omega - \omega_y = \frac{\Delta\Omega_0^2}{\Delta\Omega_0 - \Delta\omega_C/4} - \Delta\omega_C, \tag{12}$$

where $\Delta\Omega_0$ is the frequency shift for $\Delta\omega_C = 0$. The solution is valid for $|1 - 4\Delta\Omega_0/\Delta\omega_C| > 1$ while no solution exists when $|1 - 4\Delta\Omega_0/\Delta\omega_C| < 1$ and the system is Landau damped. In all cases, $Im(\Omega) \leq |Im(\Delta\Omega_0)|$, and $Im(\Omega) \ll |Im(\Delta\Omega_0)|$ when $|\Delta\Omega_0| \ll \Delta\omega_C$. Other smooth line densities give comparable results. The reason for the growth rate reduction can be understood by considering the eigenfunction $y(\phi)$. When $|\Delta\Omega_0| \ll \Delta\omega_C$ one finds that $y(\phi)$ is large only for $|\phi| \lesssim \sigma\Delta\Omega_0/\Delta\omega_C$. In physical terms, the space charge impedance causes the local coherent betatron frequency to vary along the bunch, like a collection of oscillators with different natural frequencies. The resonator is only partially effective in maintaining the oscillators at a single coherent frequency resulting in a reduced effective intensity and a subsequently reduced growth rate.

3 LANDAU DAMPING BY OCTUPOLES

Octupoles in the machine lattice give an amplitude dependent, betatron tune which contributes to the Landau damping of betatron oscillations. Chin [3] has considered this effect for a Gaussian distribution, ignoring space charge tune spread. Here, we will focus on beams with large space charge tune shift and only consider instability thresholds. As of now we have only been able to perform calculations for the case of a constant line density within the bunch. The space charge tune depression is taken to be constant within the bunch. When octupole tune spread is ignored, the eigenvalue problem can be reduced to [1]

$$b_p = -i\sum_{m=0}^{p} \frac{G_{m,p}}{Q - p + 2m} \sum_n b_n \sum_l \chi_l j_p(l\hat{\phi}) j_n(l\hat{\phi}), \tag{13}$$

where χ_l includes the space charge impedance, $j_p(x)$ is the spherical Bessel function, Q is calculated using the space charge depressed betatron frequency, $\hat{\phi}$ is the half length of the bunch, and

$$G_{m,p} = \frac{2p+1}{\pi}\frac{\Gamma(m+1/2)\Gamma(p-m+1/2)}{m!(p-m)!}.$$

Octupole induced tune spread is included by making the substitution

$$\frac{1}{Q-p+2m} \to F_\epsilon(Q-p+2m),$$

where ϵ is a measure of the betatron tune spread and $F_\epsilon(x)$ is the dispersion integral. If the distribution of the transverse action (I_y) is assumed to be $\propto (I_{max} - I_y)^2$,

$$F_\epsilon(x) = 6\int_0^1 \frac{J(1-J)dJ}{x - \epsilon J} \tag{14}$$

where $\epsilon = (\omega_y(I_{max}) - \omega_y(0))/\omega_s$. For reference, $F_\epsilon(x)$ is given by F_{1m} in Chin's notation [3]. At the instability threshold $Q = Re(Q) + i0$ and the integral in equation (14) becomes

$$\begin{aligned} F_\epsilon^t(x) &= -\frac{6}{\epsilon}\left\{r(1-r)\ln\left|\frac{1-r}{r}\right| - r + 1/2\right\} \\ &\quad -\frac{6\pi i}{|\epsilon|} r(1-r)H(r - r^2), \end{aligned}$$

where H is the Heaviside function and $r = x/\epsilon$. The threshold calculation proceeds by considering the eigenvalue problem

$$\lambda b_p = -i\sum_{m=0}^{p} G_{m,p}F_\epsilon^t(Q-p+2m)\sum_{n=0}^{K} b_n \sum_l \chi_l j_p(l\hat{\phi}) j_n(l\hat{\phi}), \tag{15}$$

where Q is fixed and real, λ is the eigenvalue, and K is the largest synchrotron mode included in the calculation. Consider the set of eigenvalues of equation (15) as Q varies, and let λ_{max} be the largest, purely real, eigenvalue. Finding λ_{max} is equivalent to finding the threshold current for the calculation. The threshold current I_{thresh} is related to the current assumed in the calculation I_{calc} via $I_{thresh} = I_{calc}/\lambda_{max}$, since substituting I_{thresh} for I_{calc} in the eigenvalue problem results in a unit eigenvalue corresponding to the smallest current for which the system is marginally unstable.

In practice, it has been found that the range $|Q| \lesssim K$ is sufficient for determining the real eigenvalues and computer code has been written to find λ_{max} for $K \leq 100$. Studies of existing machines are in progress.

4 REFERENCES

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