Quadrupolar Wakefield Induced Single-Bunch Emittance Growth in Linear Colliders

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Abstract

Emittance dilution can be induced in linear colliders by the short range quadrupolar wakefields generated, even in perfectly aligned accelerating structures, by the charge distribution of flat beams. We consider a linac with a constant phase-advance FODO lattice and we calculate, for different energy scaling laws of the beta functions along the linac, the single-bunch transverse emittance growths for a two-particle model where the bunch is represented by a head and a tail centered beam ellipse. We analyze the implication of this effect for the linar collider designs.

1 INTRODUCTION

Preserving the beam transverse emittances from being degraded along the linacs in order to reach the design luminosity at the interaction point is a major challenge of the future linear collider design. Many years ago, quadrupole wakefields were recognized as a possible source of emittance dilution [1] along the linac of the SLAC linear collider. Quadrupole wakefields are induced by the beating of the horizontal and vertical beta-functions which, even for round beams, generates a non-zero transverse charge quadrupole moment proportional to the beam averaged $\langle x^2\rangle - \langle y^2\rangle$. They in turn create a quadrupole gradient focusing error along the beam, which modifies the designed beam transport optics of the linac. However, it was soon realized that dipole wakefields are the dominant source of emittance growth. This is essentially due to the facts that, first the dipole transverse wake potential grows more rapidly with respect to the transverse coordinates (that is linearly rather than quadratically for the quadrupole one) and, second the quadrupole moment of round beams rapidly averages to zero along the linac.

In the next linear collider the obligation, related to beamstrahlung, to accelerate flat beams with a large horizontal to vertical emittance ratio reinforces the expected effect of quadrupole wakefields because the quadrupole moment, dominated by the horizontal average $\langle x^2\rangle$, remains positive along the linac. Moreover, unlike the dipole one, the quadrupole wake of a centered beam accelerated along a perfectly aligned linac does not vanish. Since the single-bunch emittance growth is a collective effect proportional to N^2, where N is the bunch population, depending in an essential way on the injection energy and on the focalisation along the linac, it is important to know what is the maximum bunch charge which can be accelerated without degrading the transverse emittance for given injection energy and focalisation optics. This is the question we want to answer in this paper by calculating the single-bunch emittance growth induced by quadrupole wakefields in the framework of the "2 particle model" where the bunch is modeled by 2 beam ellipses representing the head and the tail of the bunch. We will consider a bunch injected on-axis in a perfectly aligned linac. We will assume FODO lattice with a *constant* phase-advance and a scaling of the FODO-lengths and beta-functions as E^α, where E is the energy along the linac.

2 CALCULATION OF THE EMITTANCE GROWTH

2.1 *The 2-particle model*

We consider a bunch formed by 2 slices separated by Δz. The two slices have identical beam matrix Σ_0 at injection. While the first slice obeys the design focalisation along the linac, the second one is affected by the quadrupole gradient induced by the quadrupole wake of the first slice in the accelerating structures. Their beam matrices Σ_1 and Σ_2 therefore differ at the linac exit, although their emittances stay equal. Parametrizing a beam matrix as

$$\Sigma = \epsilon \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}$$

with $\epsilon = \sqrt{\det \Sigma}$ the emittance, the total emittance of the bunch is given by

$$\epsilon_b = \sqrt{\det\left(\frac{\Sigma_1 + \Sigma_2}{2}\right)} = \epsilon_1 \sqrt{1 + \frac{1}{4}(\delta\alpha^2 - \delta\beta\ \delta\gamma)}$$

where $\delta\alpha$, $\delta\beta$ and $\delta\gamma$ are the differences of the α, β and γ parameters between the two slices. At the lowest order, the emittance growth is then given by

$$\frac{\delta\epsilon}{\epsilon} = \frac{1}{8}(\delta\alpha^2 - \delta\beta\delta\gamma)$$

2.2 *Transverse motion with focusing error*

The transverse motion of the first slice in the linac is governed by the Hill equation with an acceleration term

$$x'' + E'/E\, x' + K\, x = 0$$

where x represents either transverse coordinate, E' is the energy gradient and K the focusing gradient along the linac. We assume that the tail slice feels a focusing gradient δK due to the quadrupole wake from the head slice in the accelerating structures [1]. Denoting by $R(s,s')$ the transfer matrix from position s' to s along the linac associated with the gradient K, and by $(R+\delta R)(s,s')$ the one associated with $K+\delta K$, one has

$$\delta\Sigma = \Sigma_2 - \Sigma_1 = \delta Q \cdot \Sigma_1 + \Sigma_1 \cdot \delta Q^{\mathsf{T}}$$

to first order in δR, with δQ the following matrix

$$\delta Q = \delta R(s_1,s_0)\cdot R^{-1}(s_1,s_0) \ ,$$

δQ^{T} its transpose, and s_0 and s_1 the beginning and end positions in the linac. The transfer matrix obeys the following differential equation

$$\partial_s R(s,s') = A(s)\cdot R(s,s')$$

with the matrix $A(s)$, deduced from Hill's equation, given by

$$A(s) = \begin{pmatrix} 0 & 1 \\ -K(s) & -(E'/E)(s) \end{pmatrix} .$$

The matrix δQ can be calculated by integrating the equation for the difference δR. Neglecting the difference $\delta(E'/E)$ in the accelerating gradient between the slices which originates from the RF-phase and the longitudinal wakefield, one gets at first order in δK

$$\delta Q = \int_{s_0}^{s_1} ds\, R(s_1,s)\cdot\delta A(s)\cdot R(s_1,s)^{-1} \ ,$$

with

$$\delta A(s) = \begin{pmatrix} 0 & 0 \\ -\delta K(s) & 0 \end{pmatrix} .$$

The above expression of δQ only involves the R_{12} and R_{22} matrix elements of $R(s_1,s)$, which can be written as

$$R_{12}(s_1,s) = \sqrt{\tfrac{E}{E_1}}\sqrt{\beta\beta_1}\sin(\Delta\psi)$$
$$R_{22}(s_1,s) = \sqrt{\tfrac{E}{E_1}}\sqrt{\tfrac{\beta}{\beta_1}}(\cos(\Delta\psi) - \alpha_1\sin(\Delta\psi))$$

where $\Delta\psi$ is the phase advance from s to s_1. This leads to the following expression of the difference between the head and tail beam matrices

$$\delta\Sigma = \epsilon_1\int_{s_0}^{s_1} ds\,\delta K(s)\,\beta(s)\begin{pmatrix} -\beta_1\sin & \alpha_1\sin-\cos \\ \alpha_1\sin-\cos & \frac{\sin}{\beta_1}(1+2\alpha_1\tan-\alpha_1^2)\end{pmatrix}$$

where the argument of the trigonometric functions is $2\Delta\psi$. The emittance growth at the lowest order in the gradient error δK is then simply given by

$$\frac{\delta\epsilon}{\epsilon} = \frac{1}{8}\left|\int_{s_0}^{s_1} ds\,\delta K(s)\,\beta(s)\,\exp(2i\Delta\psi)\right|^2 .$$

2.3 *The quadrupole wake induced focusing error*

As first discussed in [1], the quadrupole wakefields created by the first slice and averaged over a long accelerating section induce a transverse Laplace force given by

$$e(\vec{E}+\vec{v}\times\vec{B}) = N\,e^2\,w_2(\Delta z)\,Q_1\,(x\hat{x}-y\hat{y})$$

with N the bunch population, $Q_1 = \langle x^2\rangle - \langle y^2\rangle$ the transverse quadrupole moment of the head slice, $\hat{x}$ and $\hat{y}$ the unit vectors in the x and y directions, and w_2 the average quadrupolar wake potential [2] of the accelerating structure, assumed axisymmetric, per unit length and unit charge. As in [1], we assume that there is no source of xy-coupling in the linac and we neglect the effect of the skew-quadrupolar wakefields proportional to $\langle xy\rangle$. For flat beams, the large horizontal over vertical emittance ratio allows the following approximation

$$Q_1(s) = \epsilon_x(s)\,\beta_x(s) - \epsilon_y(s)\,\beta_y(s) \simeq \epsilon_x(s)\,\beta_x(s)$$

Translated in the equation of motion, the above Laplace force gives rise to the following focusing gradient

$$\delta K(s) = -\frac{N\,e^2}{E(s)}\,w_2(\Delta z)\,\epsilon_x(s)\,\beta_x(s)$$

which we parametrize as $\delta K(s) = -A(s)\,\beta_x(s)$ for later convenience. Notice that the coefficient $A(s)$ scales like the inverse of the energy squared along the linac.

2.4 *Emittance Growth after one FODO cell*

The emittance growth after one FODO cell can be easily calculated by assuming that the energy is constant over the cell and that the beam is matched. In fact, we consider a cell with μ phase-advance starting by either half a focusing quadrupole F/2 or half a defocusing one D/2. The horizontal and vertical emittance growths are then given by

$$\frac{\delta\epsilon_{x,y}}{\epsilon_{x,y}} = c_{x,y}^2\,\frac{A^2\,L^6}{2}$$

where L is the distance between two quadrupoles. The coefficients c_x and c_y depend on the phase advance and on the cell type. As shown in Table 1 for 60° and 90° phase advances, they are always of the order of one.

Table 1: Coefficients $c_{x,y}$ for the single cell emittance growth.

μ	c_x at F/2	c_y at F/2	c_x at D/2	c_y at D/2
60°	3.37	-1.73	4.63	-1.25
90°	0.727	-1.66	3.27	-1.00

2.5 *Emittance Growth over the linac*

To calculate the emittance growth over the linac, we assume that the phase advance per cell μ is constant along the linac while the length L scales as $L_0(E/E_0)^\alpha$, E_0 being the injection energy and L_0 the distance between two quadrupoles in the first cell. To simplify the calculation, we also assume that the linear rise of the energy along the linac is slow compared to the beta-wavelength

$$E'/E \ll 1/2L \ .$$

The energy can then be approximated by a step function such that it is constant and the beam is assumed to

Table 2: 500 GeV c.m. energy linear collider parameters and characteristic length L_c for $E_0 = 5$ GeV.

	CLIC	VLEPP	NLC	SBLC	TESLA
f_{RF} [GHz]	30	14	11.4	3	1.3
N [10^9]	6	200	6.5	29	50
$\gamma\epsilon_x$ [10^{-6}m]	1.8	20	5	10	20
σ_z [μm]	170	750	100	500	1000
$W(2\sigma_z)$ [cm^{-5}]	4×10^4	560	200	0.3	
$w_2(2\sigma_z)$ [V/C m]	3.6×10^{24}	5.0×10^{22}	1.8×10^{22}	2.7×10^{19}	5.0×10^{16}
L_c [m]	20	12	81	340	1830

be matched in every cell. Since the beta functions scale like the length L, the gradient error δK then scales as $\delta K_0(E/E_0)^{\alpha-2}$ where δK_0 is the gradient error in the first cell. The integral over the linac entering in the expression of the emittance growth can then be replaced by a sum over the N FODO cells composing the linac, leading to

$$\frac{\delta\epsilon}{\epsilon} = \frac{1}{8}\left|\sum_{n=1}^{N} e^{2in\mu}\left(\frac{E_n}{E_0}\right)^{3\alpha-2}\int_{s_0}^{s_0+2L_0} ds\ \delta K_0(s)\,\beta_0(s)\,e^{2i\Delta\psi}\right|^2 .$$

The emittance growth over the linac is therefore equal to the emittance growth after the first cell $(\delta\epsilon/\epsilon)_0$ times a correction factor as given by

$$\frac{\delta\epsilon}{\epsilon} = \left(\frac{\delta\epsilon}{\epsilon}\right)_0\left|\sum_{n=1}^{N} e^{2in\mu}\left(\frac{E_n}{E_0}\right)^{3\alpha-2}\right|^2 .$$

For an infinite number of cells, the above alternating sum diverges when $\alpha > 2/3$, oscillates when $\alpha = 2/3$, and converges when $\alpha < 2/3$. This clearly shows the special role of the $\alpha = 2/3$ scaling, independently of the value of the phase advance μ. For finite N, the sum can be easily evaluated when μ is a submultiple of 2π. In particular, for 90° phase advance, writing

$$\left(\frac{E_{n+1}}{E_0}\right)^{3\alpha-2} - \left(\frac{E_n}{E_0}\right)^{3\alpha-2} \simeq (3\alpha-2)\frac{2L_nE'}{E_0}\left(\frac{E_n}{E_0}\right)^{3\alpha-3}$$

and then replacing the sum over the cells back to an integral over the linac, leads to

$$\frac{\delta\epsilon_{x,y}}{\epsilon_{x,y}} = c_{x,y}^2\,\frac{A_0^2L_0^6}{8}\begin{cases}\left(\left(\frac{E_1}{E_0}\right)^{3\alpha-2}-1\right)^2 & \text{if } \alpha \neq \frac{2}{3}\\ (1-(-1)^N)^2 & \text{if } \alpha = \frac{2}{3}\end{cases}$$

where E_1 is the beam energy at the end of the linac. When $E_1 \gg E_0$, the emittance growth scales like $(E_1/E_0)^{6\alpha-4}$ if $\alpha > 2/3$, and is independent of E_1 if $\alpha \leq 2/3$. From the above discussion on the $N \to \infty$ convergence, the critical exponent 2/3 does not depend on the value of the phase advance. In fact the emittance growth for 60° phase advance is given by the 90° result divided by $\sin^2(\pi/3)$.

3 IMPLICATION FOR LINEAR COLLIDER DESIGNS

To evaluate the magnitude of this effect for the linear collider designs, we introduce the characteristic length L_c as

$$L_c = A_0^{-1/3} = \left(\frac{N\,e^2w_2(\Delta z)\,\epsilon_x(s_0)}{E_0}\right)^{-1/3} .$$

The emittance growth induced by quadrupole wakefields then reads

$$\frac{\delta\epsilon_{x,y}}{\epsilon_{x,y}} = \frac{c_{x,y}^2}{8}\left(\frac{L_0}{L_c}\right)^6\begin{cases}\left(\left(\frac{E_1}{E_0}\right)^{3\alpha-2}-1\right)^2 & \text{if } \alpha \neq \frac{2}{3}\\ (1-(-1)^N)^2 & \text{if } \alpha = \frac{2}{3}\end{cases} .$$

so that it is small only when $L_0 \ll L_c$. The length L_c is given in Table 2 for all linear colliders (except JLC for which the RF frequency is not yet known) assuming an injection energy $E_0 = 5$ GeV. The wake potential w_2, evaluated for $\Delta z = 2\sigma_z$, has been calculated with TBCI [2] for TESLA and, for the other designs, derived from the wake function $W = 4\pi\epsilon_0 w_2$ given in [1] for the SLC 3 GHz cavities down to a distance of 3 mm. For RF frequencies f_{RF} higher than 3 GHz, we used the following scaling formula

$$W^{(f_{RF})}(2\sigma_z) = \lambda^5\,W^{(3\,\mathrm{GHz})}(2\,\lambda\,\sigma_z)$$

with $\lambda = f_{RF}/3$ GHz.

As shown by Table 2, the emittance growth can be sizeable for the designs with the highest RF frequency. This is even more true if the linac optics scales linearly with energy ($\alpha = 1$) due to the correction factor $(E_1/E_0)^2$.

4 CONCLUSION

Using a simple "2 slice model", we have estimated at the lowest order in the wake potential the emittance growth induced by the transverse quadrupole wakefields for a flat bunch accelerated in a perfectly aligned linac. The emittance growth increases quadratically with the population and the horizontal emittance of the bunch. It depends also strongly on the injection energy E_0 and the linac focalisation optics. Assuming that the half beta-wavelength is given by $L_0(E/E_0)^\alpha$ as a function of the energy E along the linac, the emittance growth is proportional to $(L_0/L_c)^6$ with a correction factor $(E_1/E_0)^{6\alpha-4}$ when $\alpha > 2/3$. The characteristic length L_c sets a lower limit on the beta-wavelength close to 20 meters, and even below for $\alpha = 1$, for the highest RF-frequency linear collider designs.

5 REFERENCES

[1] A.W. Chao and R.K. Cooper, "Transverse Quadrupole Wake Field Effects in High Intensity Linacs", Particle Accelerators, vol. 13, pp. 1-12, 1983

[2] T. Weiland, "Comment on Wake Field Computation in Time Domain", NIM, vol. 216, pp. 31-34, 1983