Coupling between the Transverse and Longitudinal Motion in an AVF Racetrack Microtron

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Abstract

A study is made of racetrack microtrons of which the bending magnets have a small azimuthally varying field (AVF) profile superimposed on the average main magnetic field. Two such microtrons are under construction at the Eindhoven University of Technology. In this paper the effect of the AVF profile on the coupling between transverse and longitudinal motion in AVF racetrack microtrons is studied. Stability conditions are derived and a comparison with the uncoupled case is made.

1 INTRODUCTION

The description presented in this paper is a summary of work reported in reference [1].

The principle of applying an AVF profile on the bending magnets of a racetrack microtron has been described to a great extent in previous work, e.g. [2, 3]. In this section, details relevant for the present paper are summarized.

The origin of a polar coordinate system (r, θ, z) is positioned at the point where a reference particle enters the (righthand) dipole magnet of the racetrack microtron, see Fig. 1. The vertical component of the magnetic field in the median plane of an AVF magnet consists of a main magnetic field, B_0, and an azimuth-dependent flutter profile, $f(\theta)$, i.e.

$$B_z = B_0[1 + f(\theta)], \quad f(0) = 0.$$

All results presented in this paper are accurate up to first order in (the amplitude of) the flutter profile. Such a first-order description suffices if the following condition is satisfied

$$\int_0^{\pi/2} \frac{|f(\xi)|}{\sin^2(\xi)} d\xi \ll 1.$$

The reference trajectory $r(\theta)$ through the AVF magnet can be calculated to be

$$r(\theta) = R\left(2\sin\theta - \frac{2}{\sin\theta}\int_0^{\theta} f(\xi)\sin(2\theta - 2\xi)d\xi\right), \quad (1)$$

where $R = P_{\mathrm{ref}}/eB_0$, with P_{ref} the linear momentum of the reference particle and e the elementary charge. So, R is the radius of the 'unperturbed' circular orbit. The above equation for the reference trajectory suffices to compute the orbital length, s, and the exit angle, Ψ. The latter is given by

$$\Psi = -2\int_0^{\pi/2} f(\xi)\cos(2\xi)d\xi. \quad (2)$$

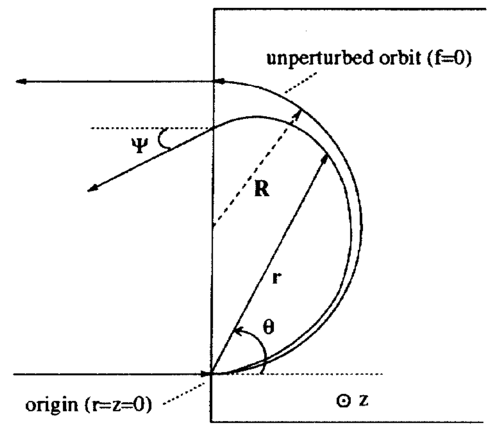

Figure 1: Overview of the magnet geometry.

2 RELATIVE MOTION

Consider the motion of an electron in a tube around the reference orbit. In this tube, we introduce a curvilinear coordinate system x, s, z, where s is the orbital length of the point on the reference orbit which is nearest to the electron. The coordinate x gives the deviation perpendicular to the reference orbit in the median plane and the coordinate z the deviation perpendicular to the median plane. The Hamiltonian in this coordinate system takes the form

$$\begin{aligned}\mathcal{H}_1 \;=\; & \left[E_0^2 + (p_s/\{1 + x/\rho(s)\} + eA_s)^2c^2\right. \\ & \left.+(p_x + eA_x)^2c^2 + (p_z + eA_z)^2c^2\right]^{1/2},\end{aligned}$$

where $\vec{p} = p_x\vec{e}_x + p_s\vec{e}_s + p_z\vec{e}_z$ is the canonical momentum, $\vec{A} = A_x\vec{e}_x + A_s\vec{e}_s + A_z\vec{e}_z$ the vector potential, E_0 the electron rest energy and c the velocity of light.

Since a general particle is assumed to deviate little from the reference particle, we linearize the equations of motion at every time t around the position of the reference particle ($s = s_{\mathrm{ref}}(t)$, $x = z = 0$). Note that we also have to linearize the coordinate system. We introduce small coordinates $\hat{x}, \hat{s}, \hat{z}$ according to

$$x = \hat{x}, \quad s = s_{\mathrm{ref}}(t) + \hat{s}, \quad z = \hat{z},$$
$$p_x = \hat{p}_x, \quad p_s = P_{\mathrm{ref}}(t) + \hat{p}_s, \quad p_z = \hat{p}_z.$$

The relevant part of the vector potential at a general position, expressed in terms of the magnetic field and the coordinate system at the reference position, can be chosen as

$$\vec{A} = \left\{\hat{x}B_z + \frac{\hat{x}^2}{2}\left(\frac{\partial B_z}{\partial x} - \frac{B_z}{\rho}\right) - \frac{\hat{z}^2}{2}\frac{\partial B_z}{\partial x} + \hat{x}\hat{s}\frac{\partial B_z}{\partial s}\right\}\vec{e}_s.$$

In order to get dimensionless quantities, a scaling transformation is applied

$$\tilde{x} = \hat{x}/R, \quad \tilde{s} = \hat{s}/R, \quad \tilde{z} = \hat{z}/R, \quad \tilde{t} = ct/R,$$
$$\tilde{p}_x = \hat{p}_x/P_{\text{ref}}, \quad \tilde{p}_s = \hat{p}_s/P_{\text{ref}}, \quad \tilde{p}_z = \hat{p}_z/P_{\text{ref}}.$$

Since the magnetic field is only θ-dependent, we take θ as independent variable instead of $\tilde{t}$. From the expression for the reference orbit, Eq. (1), the following relation between $\tilde{t}$ and θ can be found (assuming $\beta = 1$)

$$d\tilde{t} = 2[1 - F(\theta)]d\theta, \quad F(\theta) = \frac{1}{\sin^2\theta}\int_0^\theta f(\xi)\sin(2\xi)d\xi.$$

This results in the following Hamiltonian with θ as independent variable, already expanded up to second degree in the canonical variables

$$\mathcal{H}_2 = [1 - F(\theta)]\left\{\tilde{p}_x^2 + \tilde{Q}_x\tilde{x}^2 - 2\tilde{b}\tilde{x}\tilde{p}_s + \tilde{p}_z^2 + \tilde{Q}_z\tilde{z}^2\right\},$$

where

$$\tilde{Q}_x(\theta) = 1 + 2f(\theta) - \frac{1}{2\tan(\theta)}\frac{df}{d\theta} =: 1 + \bar{Q}_x(\theta),$$

$$\tilde{Q}_z(\theta) = \frac{1}{2\tan(\theta)}\frac{df}{d\theta}, \quad \tilde{b}(\theta) = 1 + f(\theta),$$

all valid up to first order in f. From the Hamiltonian we see that the vertical motion is decoupled from the motion in the median plane. Therefore, the vertical motion can be solved independently and will not be considered in this paper, cf. [3]. In order to decouple the horizontal and longitudinal motion up to zero order (note the $\tilde{x}\tilde{p}_s$ coupling term), the following transformation is applied

$$\bar{x} = \tilde{x} - \tilde{p}_s, \quad \bar{p}_x = \tilde{p}_x, \quad \bar{s} = \tilde{s} - \tilde{p}_x, \quad \bar{p}_s = \tilde{p}_s.$$

The resulting Hamiltonian gives rise to a system of differential equations that can be written as

$$d\vec{X}/d\theta = [A + B(\theta)]\vec{X}, \quad \vec{X} = (\bar{x}, \bar{p}_x, \bar{s}, \bar{p}_s)^{\mathrm{T}},$$

where A is the zero-order matrix

$$A = \left(\begin{array}{cc|cc} 0 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

and the non-zero elements of B (all first order) are

$$B_{12}(\theta) = -2F(\theta), \quad B_{21}(\theta) = 2[F(\theta) - \bar{Q}_x(\theta)],$$
$$B_{24}(\theta) = 2[f(\theta) - \bar{Q}_x(\theta)], \quad B_{31}(\theta) = -B_{24}(\theta),$$
$$B_{34}(\theta) = 2[F(\theta) - 2f(\theta) + \bar{Q}_x(\theta)].$$

This system is equivalent to the Volterra integral equation

$$\vec{X}(\theta) = e^{\theta A}\vec{X}(0) + \int_0^\theta e^{(\theta-\xi)A}B(\xi)\vec{X}(\xi)d\xi.$$

The solution to this equation can be found by successive substitution. Starting with the initial approximation $\vec{X} = \vec{0}$, the transition matrix up to first order in f turns out to be

$$e^{\theta A} + e^{\theta A}\int_0^\theta e^{-\xi A}B(\xi)e^{\xi A}d\xi.$$

Transforming this transfer matrix back to the physical tilde-variables gives as the final solution

$$\vec{Y}(\theta) = [U(\theta) + V(\theta)]\vec{Y}(0), \quad \vec{Y} = (\tilde{x}, \tilde{p}_x, \tilde{s}, \tilde{p}_s)^{\mathrm{T}},$$

with $U(\theta) =$

$$\left(\begin{array}{cc|cc} \cos(2\theta) & \sin(2\theta) & 0 & 1 - \cos(2\theta) \\ -\sin(2\theta) & \cos(2\theta) & 0 & \sin(2\theta) \\ \hline -\sin(2\theta) & -1 + \cos(2\theta) & 1 & -2\theta + \sin(2\theta) \\ 0 & 0 & 0 & 1 \end{array}\right),$$

and the non-zero elements of V are given by

$$\begin{aligned}
V_{11}(\theta) &= J_{11}(\theta)C + J_{21}(\theta)S, \\
V_{12}(\theta) &= J_{12}(\theta)C + J_{22}(\theta)S, \\
V_{14}(\theta) &= J_{14}(\theta)C + J_{24}(\theta)S - J_{11}(\theta)C - J_{21}(\theta)S, \\
V_{21}(\theta) &= -J_{11}(\theta)S + J_{21}(\theta)C, \\
V_{22}(\theta) &= -J_{12}(\theta)S + J_{22}(\theta)C, \\
V_{24}(\theta) &= -J_{14}(\theta)S + J_{24}(\theta)C + J_{11}(\theta)S - J_{21}(\theta)C, \\
V_{31}(\theta) &= -J_{11}(\theta)S + J_{21}(\theta)C + J_{31}(\theta), \\
V_{32}(\theta) &= -J_{12}(\theta)S + J_{22}(\theta)C + J_{32}(\theta), \\
V_{34}(\theta) &= -J_{14}(\theta)S + J_{24}(\theta)C + J_{11}(\theta)S - J_{21}(\theta)C + \\
&\quad J_{34}(\theta) - J_{31}(\theta),
\end{aligned}$$

where $S = \sin(2\theta)$, $C = \cos(2\theta)$, and

$$\begin{aligned}
J_{11}(\theta) &= \int_0^\theta \bar{Q}_x(\xi)\sin(4\xi)d\xi, \\
J_{12}(\theta) &= \int_0^\theta \left\{\bar{Q}_x(\xi)[1 - \cos^2(4\xi)] - 2F(\xi)\right\}d\xi, \\
J_{14}(\theta) &= -\int_0^\theta B_{24}(\xi)\sin(2\xi)d\xi, \\
J_{21}(\theta) &= \int_0^\theta \left\{-\bar{Q}_x(\xi)[1 + \cos^2(4\xi)] + 2F(\xi)\right\}d\xi, \\
J_{22}(\theta) &= -J_{11}(\theta), \\
J_{24}(\theta) &= \int_0^\theta B_{24}(\xi)\cos(2\xi)d\xi, \\
J_{31}(\theta) &= -J_{24}(\theta), \\
J_{32}(\theta) &= J_{14}(\theta), \\
J_{34}(\theta) &= \int_0^\theta B_{34}(\xi)d\xi.
\end{aligned}$$

The solution $(U + V)$ describes the evolution of the deviation-vector $\vec{Y}$ as a function of θ in a single magnet. In practice, one is also interested in the transfer matrix for a complete revolution through the racetrack microtron. In this case, the reference orbit needs to be closed. For a general profile $f(\theta)$ this may not be the case, since the exit angle Ψ (see Eq. (2)) may be unequal to zero. In order to assure that the orbits are actually closed, even if $\Psi \neq 0$,

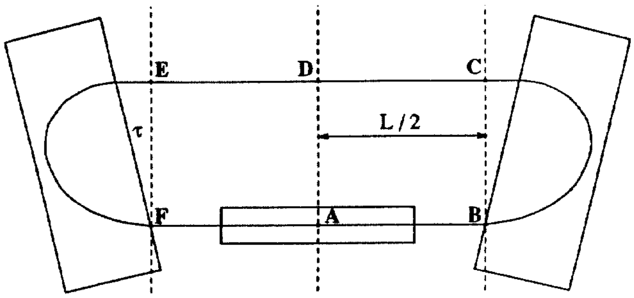

Figure 2: Depiction of one complete revolution.

the magnets are rotated through the median plane over the angle $\tau = \Psi/2$ (the 'tilt angle'; for details, cf. [3]). This magnet rotation gives rise to additional quadrupoles due to edge focusing effects.

3 STABILITY CONDITIONS

Now, to obtain the transfer matrix for a full revolution, the following separate transfer matrices need to be multiplied (starting from the centre of the accelerating cavity, point A in Fig. 2): (i) half the driftspace, including the righthand half of the cavity (A to B); (ii) the righthand rotated AVF magnet (B to C); (iii) the full driftspace (C to E); (iv) the lefthand rotated AVF magnet in reversed direction (E to F); and finally (v) half the driftspace, including the lefthand half of the cavity (F to A). In the horizontal phase plane, the cavity is assumed to have no effect (i.e.: the adiabatic damping and the weak focusing lens are neglected); in the longitudinal phase plane, the well-known effect of a phase-dependent energy gain (linearized around the synchronous phase ϕ_s) is taken into account. We write the transition matrix for one complete revolution (A to A) as

$$\vec{Y}_{A,1} = N_{xs}\vec{Y}_{A,0} = \left(\begin{array}{c|c} N_{xs,11} & N_{xs,12} \\ \hline N_{xs,21} & N_{xs,22} \end{array}\right) \vec{Y}_{A,0},$$

where $N_{xs,ij}$ are 2×2 matrices. After some computations it turns out that $N_{xs,12} = \mathcal{O}$. This implies that the eigenvalues of N_{xs} (which are decisive for beam stability) are equal to the eigenvalues of $N_{xs,11}$ and $N_{xs,22}$. Since the determinant of both these matrices equals unity (adiabatic damping neglected), the stability of the motion is determined by the well-known trace-criterion:

$$|\mathrm{Tr}(N_{xs,11})| < 2, \quad |\mathrm{Tr}(N_{xs,22})| < 2.$$

This yields

$$0 < \bar{a} + 4\tau < 2R/L, \tag{3}$$

$$0 < \tan(\phi_s) < 2/\pi\nu, \tag{4}$$

where ν is the incremental harmonic number (pathlength difference between two successive orbits in the racetrack microtron, divided by the cavity resonant wavelength), and

$$\bar{a} = -2\int_0^{\pi/2} \frac{f(\xi)}{\sin^2(\xi)} d\xi.$$

Note that these conditions are conditions for the stability of one orbit (i.e. one value of R).

In Ineq. (4), the path lengthening due to the flutter profile has been incorporated explicitely by demanding that ν remains fixed (integer). This can be accomplished by a slight change of the main magnetic induction in the bending magnets, B_0. As a result, Ineq. (4) is exactly equal to the well-known longitudinal stability condition for microtrons. Apparently, the presence of the AVF profile has not altered the synchrotron tune, although the local phase/energy oscillation amplitudes will be changed.

Also Ineq. (3) is exactly equal to the result obtained in previous work [3], where the energy spread and longitudinal dimension of the beam were neglected. The horizontal stability condition (3) needs to be combined with the stability condition for vertical motion [3] before an ideal AVF profile can be determined. Since the horizontal stability condition is not altered by the transverse/longitudinal coupling (although local betatron oscillation amplitudes will have changed), results obtained in previous work are still valid. Hence, the magnet rotation remains an essential ingredient for the obtainment of transverse beam stability in an AVF racetrack microtron.

It was found that $N_{xs,12} = \mathcal{O}$. This means that the horizontal motion for one revolution from the middle of the cavity to the middle of the cavity does not depend on the longitudinal motion. So, for the horizontal motion, it suffices to consider an eigenellipse in a two-dimensional phase plane. However, the longitudinal motion for one revolution depends on the horizontal motion ($N_{xs,21} \neq \mathcal{O}$), so for the stability of the longitudinal motion, it is necessary to consider the eigenellipsoid in four-dimensional phase space.

One final remark. We examined stability of a single revolution through the racetrack microtron. For this, we considered the motion from the middle of the cavity to the middle of the cavity and saw that the particle deviations can be kept within certain bounds at this point. From the description presented, it cannot be estimated how large the deviations will become at other points of the orbit — it is difficult to compute the beta-function as a function of orbit length analytically. Additionally, the eigenellipsoids change with increasing R. Therefore, it is necessary to calculate the beam sizes and machine acceptance numerically (using the transfer matrices in this paper) so as to determine the best AVF profile.

4 REFERENCES

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