# Simulation of 4-Turn Algorithms for Reconstructing Lattice Optic Functions from Orbit Measurements

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*Abstract*

We describe algorithms for reconstructing tune, closed-orbit, beta-function and phase advance from four individual turns of beam orbit acquisition data, under the assumption of coherent, almost linear and uncoupled betatron oscillations. To estimate the beta-function at, and phase advance between, position monitors, we require at least one anchor location consisting of two monitors separated by a drift. The algorithms were submitted to a Monte Carlo analysis to find the likely measurement accuracy of the optics functions in the KAON Factory Booster ring racetrack lattice, assuming beam position monitors with surveying and reading errors, and assuming an imperfect lattice with gradient and surveying errors. Some of the results of this study[2] are reported.

## 1 INTRODUCTION

The primary motivation for studying the 4-turn algorithms was the anticipated poor measurement accuracy of beam related variables in a fast-cycling machine. In a synchrotron, the lattice optical properties must be measured and corrected at several time intervals during the magnet field ramp because tracking errors will cause them to vary. Conventional optics measurement techniques (e.g. tune vs. quadrupole strength), as used at storage rings, are inappropriate for a rapid cycling (50Hz) synchrotron where the optics may change during a lengthy measurement.

### *1.1 Operating modes of KAON Booster*

The proposed TRIUMF KAON Factory has recently been cancelled. The Booster[1] accelerator design included lattices with achromatic arcs and long dispersionless straight sections. Dispersion is suppressed by choosing the tune of each arc to be an integer. The overall horizontal tune, is set using the straight sections. There are two high current modes, 2nd and 1st order arc achromats. There is also a polarized running mode in which the optics is re-tuned to give spin transparent straights. The closed orbit will have to be corrected for each operating mode. Further, the focusing strength has to change considerably, which implies possibly severe distortion of the beta-function – unless the powerful trim families are carefully controlled.

Tuning the optics requires setting and measuring partial tunes across arcs and straights, and careful measurement and control of beta-function. For second order achromat running, partial tune errors of $|\delta\nu|^{\rm arc} = 0.01$ per arc are not significantly damaging. For polarized running, partial tune errors of $|\delta\nu_y|^{\rm s.s.} = 0.01$ per straight are tolerable. The betatron tune should be controlled to $|\delta\nu| \leq 0.01$, and measured to $|\delta\nu| \leq 0.0025$.

### *1.2 Single turn orbit acquisition*

The diagnostic technique of single turn beam orbit acquisition means the ability to record a snap-shot of the beam transverse displacement at many locations around the ring over the course of a turn, and to do this for several turns. This allows the possibility for rapid tune and closed orbit measurements, and for reconstruction of the beta-function without varying individual quadrupole strengths.

## 2 4-TURN ALGORITHMS

If the position monitor is error-free, then the turn-by-turn position is $x_n = X_{\rm c.o.d.} + A\cos[\phi + n\mu] + D\,\Delta p(n)/p_0$ for the beam centroid. Here $\mu = 2\pi\nu$ is the phase advance per turn, $A = \sqrt{\beta}$ and $\phi$ are the betatron amplitude and phase at the particular monitor; subscript $n$ indicates the turn number.

<u>4-turn tune estimate</u>

Following Risselada[3] algebraic manipulation gives the 4-turn tune estimate at each position monitor as:

$$\cos\mu = [(x_2 - x_1) + (x_4 - x_3)]/2(x_3 - x_2) \ ,$$

which is not biased by any systematic monitor error. An analytic estimate of the accuracy is given in Reference[2].

<u>4-turn c.o.d. estimate</u>

The closed orbit estimate at each monitor is given by

$$X_{\rm c.o.d} = [x_3(x_1 + x_3) - x_2(x_2 + x_4)]/[(x_1 - x_4) + 3(x_3 - x_2)] \ .$$

This algorithm does not reject systematic monitor error, and requires an on-momentum beam.

### *2.1 Monte Carlo analysis*

The 4-turn measurement has the advantage that it is quick; and the variation of closed orbit and tune over such a short time period should be very small. Some penalty must be associated with making inferences from less data, and so a Monte Carlo error analysis is reported below.

<u>Machine errors</u>

The following statistical error model is used for the perturbed lattice: $\pm 0.25$ mm r.m.s. and $\pm 0.50$ mm r.m.s. transverse displacement for bends and quadrupoles, respectively, in horizontal and vertical planes; $\pm 0.50$ mm r.m.s. longitudinal displacement, $\pm 1.0$ mrad r.m.s. and $\pm 10^{-3}$ roll and relative fractional strength, respectively, for all elements; $\pm 1.0$ mrad r.m.s. tilt for bends.

BPM errors

The Beam Position Monitors (BPMs) are taken to have systematic errors and random errors. For a particular BPM, the systematic error is the same every time we read that monitor; but the random error may change each time we read it. We take systematic errors with r.m.s. deviation $\sigma_s = 0.75$ mm and random errors with $\sigma_r = 0.25$ mm; and both errors to be truncated at $3 \times \sigma$ . The total number of BPMs was 30 in each plane, horizontal and vertical.

We considered an ideal lattice with fractional tunes $\nu_x = 0.35$ (or 0.650), $\nu_y = 0.40$ (or 0.60), and a perturbed lattice with fractional tunes $\nu_x = 0.35085$, $\nu_y = 0.38533$ . A statistical analysis of 20 different perturbed lattices, generated with DIMAD, showed this to be a typical lattice.

### 2.2 *Indication of c.o.d. measurement accuracy*

We attempted to find how accurately the 4-turn algorithm would predict the closed orbit distortion, assuming an on-momentum beam. We took an ideal lattice for which $X_{\text{c.o.d.}} = 0$ everywhere. Note, it is neither desirable nor necessary to kick to measure closed orbit; however, in order to model some coherent motion (which prevents easy measurement of the c.o.d.) we chose to prepare this by kicking; results are insensitive to strengths 0.1–1.0 mrad.

From each of 60 trials (with different BPM random errors), we obtained a set of c.o.d. estimates, one per monitor location. The results are averaged over all active monitors and then averaged over all trials. We can expect, from a single trial, to estimate the true closed orbit distortion to $\pm 2$ mm. Repetition, and averaging over 3–4 trials reduces the error to $\pm 1$ mm horizontal and vertical.

### 2.3 *Indication of tune measurement accuracy*

We attempted to find how accurately the 4-turn algorithm would predict the betatron tune. The method is to take a beam initially on-axis, kick it transversely to give a coherent oscillation, and then record tracks on four turns. The c.o.d. in the perturbed lattice is corrected using BPMs with reading errors and steering magnets before the tune measurement. In order to state the 'likely' error in a typical single tune measurement, we made 60 trials.

Sextupoles off  Assuming a 1.0 mrad initial kick, the tune can be established with accuracy of $\pm 0.002$ after 1 trial. Using a weaker 0.3 mrad kick, one needs 4–5 trials to achieve the desired accuracy.

Sextupoles on  Assuming a 1.0 mrad initial kick, the tune non-linear shift can be resolved with confidence after averaging over 16–20 trials.

## 3 BETA FUNCTION AND PHASE ADVANCE 5-TURN ALGORITHM

It is possible with a single turn orbit acquisition system to reconstruct the periodical beta-function (as sampled at) and phase advance between BPMs. There is no need to know the closed orbit, nor to use an on-momentum beam. We now present an algorithm for finding ($\beta$, $\phi$); though we discovered it for ourselves, we later found it to be similar to that of Reference[4]. The principle is to find the absolute Twiss parameters at one particular location (the 'anchor'), and to propagate $\beta$ and $\phi$ to other locations.

### 3.1 *Finding a principal trajectory*

Let $C$ and $S$ be the 'principal trajectories'. For displacement $y$ and divergence $y'$ about the on-momentum closed-orbit, the values at general location $s$ downstream of $s_0$ are given by:

$$\begin{bmatrix} y \\ y' \end{bmatrix}_s = \begin{bmatrix} C(s,s_0) & S(s,s_0) \\ C'(s,s_0) & S'(s,s_0) \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}_{s_0} . \quad (1)$$

Suppose there is: (i) an anchor point, $s_0$, where we can measure both $y$ and $y'$; (ii) a position monitor at location $s$. Suppose the beam makes a coherent oscillation, and that we record $y(s_0)$, $y'(s_0)$ and $y(s)$ on two turns. The two data sets, labelled by subscripts 1 and 2, can be written as follows:

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} y_1(s_0) & y_1'(s_0) \\ y_2(s_0) & y_2'(s_0) \end{bmatrix} \begin{bmatrix} C(s,s_0) \\ S(s,s_0) \end{bmatrix} \quad (2)$$

which can be inverted for $C$ and $S$ .

The effect of closed orbit distortion $X_{\text{c.o.d.}}$ and dispersion $D(s)$ (on a beam with average off-momentum $\Delta p/p_0$)

$$y^*(s) = X_{\text{c.o.d.}} + y(s_0)C(s,s_0) + y'(s_0)S(s,s_0) + D(s)\Delta p/p_0 .$$

can be removed by taking the difference between two turns. Hence we introduce the difference quantities (3)

$$Y_{21}(s) = y_2^*(s) - y_1^*(s) \quad Y_{21}'(s) = \partial Y_{21}/\partial s . \quad (3)$$

which transform between the two locations as equation (1). If we supplement this with an equation for the differences of two more turns, then an analogue of (2) may be used to find the functions $C$ and $S$. We now use $y_1$, $y_2$ etc. as a shorthand for $Y_{21}$, $Y_{32}$ etc..

Let the length of the closed orbit be $c_0$. At the anchor we can find all elements of the transfer matrix. Let us label the data sets by the subscripts 1, 2, 3. To obtain the one-turn principal functions, we invert the equations:

$$\begin{bmatrix} y_3(s_0) \\ y_2(s_0) \end{bmatrix} = \begin{bmatrix} y_2(s_0) & y_2'(s_0) \\ y_1(s_0) & y_1'(s_0) \end{bmatrix} \begin{bmatrix} C(s_0+c_0,s_0) \\ S(s_0+c_0,s_0) \end{bmatrix} \quad (4)$$

$$\begin{bmatrix} y_3'(s_0) \\ y_2'(s_0) \end{bmatrix} = \begin{bmatrix} y_2(s_0) & y_2'(s_0) \\ y_1(s_0) & y_1'(s_0) \end{bmatrix} \begin{bmatrix} C'(s_0+c_0,s_0) \\ S'(s_0+c_0,s_0) \end{bmatrix} \quad (5)$$

The Twiss parameters are given by the identities:

$$2\cos 2\pi\nu = C + S' , \quad 2\alpha(s_0)\sin 2\pi\nu = C - S' \quad (6)$$

$$\beta(s_0)\sin 2\pi\nu = S , \quad \gamma(s_0)\sin 2\pi\nu = -C' . \quad (7)$$

Propagation of $\beta$ from the anchor to another location is performed by the identity: $\beta(s) =$

$$C^2(s,s_0)\beta(s_0) - 2C(s,s_0)S(s,s_0)\alpha(s_0) + S^2(s,s_0)\gamma(s_0) . \quad (8)$$

The phase advance $\phi(s,s_0)$ from the anchor to another location can be found from

$$S(s,s_0) = \sqrt{\beta(s)\beta(s_0)}\sin\phi(s,s_0) . \quad (9)$$

### 3.2 *Accuracy of beta and phase measurement*

We tried to estimate the accuracy with which beta-function and phase-advance can be measured by analysis of multi-track data for the KAON Booster ring. The procedure is to take an ideal lattice, add random alignment, tilt, strength, etc. errors and then (using BPMs with errors) correct the closed orbit; then for the corrected lattice make an attempt to reconstruct $\beta(s)$ and phase advance $\phi(s, s_0)$ (as sampled at monitor locations) from BPM readings with errors. It was immediately clear that a single trial could not produce the desired accuracy, and the Monte Carlo analysis is based upon averaging over 17 trials.

Anchor position monitors

The separation of the two position monitors comprising the vertical and horizontal anchors is 2.5 m and 1.5 m of drift space, respectively. The longitudinal error in the location at which an anchor BPM makes its reading is 1.0 mm r.m.s. truncated at $3 \times \sigma$.

Dependence on kick strength

We compared results for two different strengths of the initial coherent kick. For the 0.3 mrad kick, despite the large systematic error, nevertheless one could distinguish the regularity of the arc beta-function. For the 1.0 mrad kick, figure 1 shows the nicely reconstructed vertical optics functions; solid lines are exact values from DIMAD, and broken lines are measured values. For the 1.0 mrad kick, figure 2 shows the absolute phase errors and relative beta function error. The horizontal beta function is indistinguishable (by eye) from the DIMAD values.

Figures of merit, etc

We have tried to give some 'figures of merit' which typify how well an average BPM reconstructs the optics. We took the data which is already averaged over trials, and averaged the values over monitors. Hence $\langle\Delta\beta\rangle$ is the likely beta-function-error averaged over monitors and $\sigma[\Delta\beta]$ is the standard deviation about the mean error; and similarly for the other tabulated quantities.

| Case | kick (mrad) | $\langle\Delta\beta\rangle$ (m) | $\sigma[\Delta\beta]$ (m) | $\langle\Delta\beta/\beta_0\rangle$ % | $\sigma[\Delta\beta/\beta_0]$ % |
|---|---|---|---|---|---|
| Hori | 0.3 | 0.20 | 0.72 | 1.3 | 8.9 |
| Hori | 1.0 | -0.05 | 0.25 | -.45 | 2.9 |
| Vert | 0.3 | 5.7 | 4.4 | 71.6 | 70.3 |
| Vert | 1.0 | 1.1 | 0.95 | 12.7 | 5.3 |

| Case | kick (mrad) | $\langle\Delta X_{cod}\rangle$ (mm) | $\sigma[\Delta X]$ (mm) | $\langle\Delta\phi\rangle$ (deg) | $\sigma[\Delta\phi]$ (deg) |
|---|---|---|---|---|---|
| Hori | 0.3 | .015 | 0.82 | 0.69 | 23.0 |
| Hori | 1.0 | .010 | 0.82 | 0.84 | 10.8 |
| Vert | 0.3 | -.16 | 0.68 | -0.85 | 14.7 |
| Vert | 1.0 | -.18 | 0.71 | -0.64 | 4.6 |

From the r.m.s. values in the table, it is clear that the weaker kick (0.3 mrad) is not sufficient for beta and phase measurement. The strong diagnostic kick (1.0 mrad) is satisfactory, though the peak errors in phase advance are disturbingly large (Fig.2). The errors could be reduced by increasing the number of anchors and/or by developing some method to better reject outlying observations. Note, the accuracy of a single trial is inadequate, and usually one must average over something like 6–8 trials.

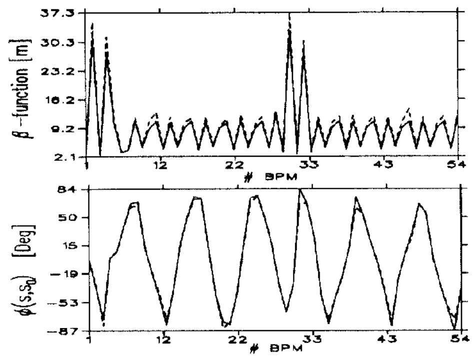


Figure 1: Measured vertical lattice functions.

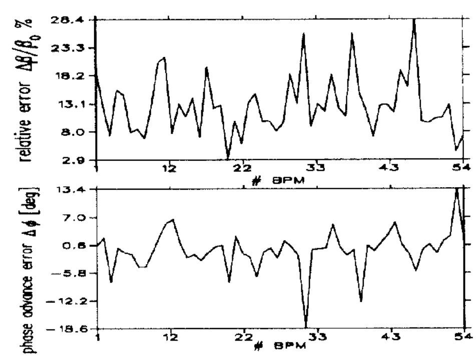


Figure 2: Error in measured vertical lattice functions.

## 4 CONCLUSION

For the KAON Booster ring we have studied by Monte Carlo simulation the accuracy of methods for obtaining bare tune, closed orbit, (coherent) beta-function and partial tune of straights and arcs from analysis of beam tracks. A single 4-turn measurement does not provide adequate measurement accuracy. With 16 turns of memory for every beam position monitor, it is possible to measure the overall tune to $\pm 0.002$ (or better) for a lattice with imperfections and chromaticity correction. The same system can measure the closed orbit to $\pm 1.0$ mm. This is against a background of systematic errors 0.75 mm r.m.s. and random error 0.25 mm r.m.s.. Measurement of beta-function and phase-advance required a diagnostic kick strength of 1.0 mrad to achieve acceptable accuracy. A relative accuracy $\Delta\beta/\beta_0$ of 3–5 % and absolute accuracy $\Delta\phi$ of 5–10 degrees can be obtained from averaging over 16 turns of orbit acquisition.

## 5 REFERENCES


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