Symplectic Tracking Using Point Magnets in the Presence of a Longitudinal Magnetic Field*

G. Parzen

Brookhaven National Laboratory
Upton, NY 11973, USA

Abstract

Symplectic tracking with point magnets in the presence of a longitudinal field is achieved by replacing the drift spaces by a longitudinal drift, which is the motion of a particle in a uniform longitudinal field. Results are given for the transfer functions.

1 INTRODUCTION

In the absence of a longitudinal magnetic field, symplectic tracking can be achieved by replacing the magnets [1] by a series of point magnets and drift spaces. To treat the case when a longitudinal magnetic field is also present, this procedure is modified in this paper by replacing the drift space by a solenoidal drift, which is defined as the motion of a particle in a uniform longitudinal magnetic field. A symplectic integrator can be obtained by subdividing each magnet into pieces and replacing each magnet piece by point magnets, with only transverse fields, and solenoidal drift spaces. The reference orbit used here is made up of arcs of circles and straight lines [2] which join smoothly with each other. For this choice of reference orbit, the required results are obtained to track particles, which are the transfer functions, and the transfer time for the different elements. It is shown that these results provide a symplectic integrator, and they are exact in the sense that as the number of magnet pieces is increased, the particle motion will converge to the particle motion of the exact equations of motion.

2 THE APPROXIMATE LATTICE

In the absence of a longitudinal magnetic field, one procedure for symplectic integration is to replace each magnet in the given lattice by a series of point magnets and drift spaces. The equations of motion for the approximate lattice which has only point magnets and drifts can be integrated exactly, which gives a symplectic second order integrator [2] for the case where the longitudinal magnetic field, B_s, is absent.

For the case where a longitudinal magnetic field is present, $B_s \neq 0$, the following approximate lattice is proposed. Each magnet is broken into a number of pieces. A magnet piece of length h is replaced by point magnets, at each end of the piece with only transverse fields B_x, B_y, and a solenoidal drift which is defined as the particle motion in a uniform longitudinal magnetic field. The point magnets at the ends of the piece, kick the values of p_x and p_y at each end of the piece. In between the point magnets, the particle performs a solenoidal drift; the particle coordinates change as they would in a uniform longitudinal magnetic field.

It will be shown below that the above proposed approximate lattice for the case when $B_s \neq 0$ gives a symplectic integrator. This integrator is correct to first order in h, using the simplest procedure for specifying the longitudinal field in the solenoidal drift. More complicated procedures for specifying the longitudinal field may improve the accuracy. However first order in h accuracy may be sufficient as the effects due to the longitudinal fields are often small. As one increases the number of magnet pieces, decreasing h, the result obtained by integrating this approximate lattice will converge to the actual motion for the given lattice.

3 THE EQUATIONS OF MOTION

The equations of motion for the transverse coordinates may be written as [3]

$$\begin{aligned}
\frac{dx}{ds} &= \frac{1+x/\rho}{p_s} p_x \\
\frac{dp_x}{ds} &= \frac{p_s}{\rho} + \frac{e}{c}\left[(1+x/\rho)\,B_y - \frac{p_y}{p_s}(1+x/\rho)\,B_s\right] \\
\frac{dy}{ds} &= \frac{1+x/\rho}{p_s} p_y \\
\frac{dp_y}{ds} &= \frac{e}{c}\left[\frac{p_x}{p_s}(1+x/\rho)\,B_s - (1+x/\rho)\,B_x\right] \\
p_s &= \left(p^2 - p_x^2 - p_y^2\right)^{1/2}
\end{aligned} \tag{3.1a}$$

x, y are the transverse coordinates in a coordinate system based on a reference orbit with the radius of curvature $\rho(s)$. As the longitudinal coordinates one can use t, the particle time of arrival at s, and E the particle energy. The longitudinal coordinates obey the equations

$$\begin{aligned}
\frac{dt}{ds} &= \frac{1+x/\rho}{p_s}\frac{p}{v} \\
\frac{dE}{ds} &= e\,(1+x/\rho)\,\mathcal{E}_s
\end{aligned} \tag{3.1b}$$

In Eq. (3.1) it has been assumed that the electric field has only the longitudinal component, $\mathcal{E}_s$. One can show that the equation for dt/ds is equivalent to

$$\begin{aligned}
dt &= \frac{d\ell}{v} \\
d\ell &= \left[(1+x/\rho)^2 + (dx/ds)^2 + (dy/ds)^2\right]^{1/2} ds
\end{aligned} \tag{3.1c}$$

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where $d\ell$ is the path length over ds.

The equations of motion from Eqs. (3.1) may be derived from the hamiltonian [3]

$$H = -(1+x/\rho)\left[E^2/c^2 - m^2c^2 - (\Pi_x - eA_x/c)^2 - (\Pi_y - eA_y/c)^2\right]^{1/2} - e(1+x/\rho)A_s/c \tag{3.2a}$$

where Π_x, Π_y, the coordinates canonical to x, y are

$$\begin{aligned} \Pi_x &= p_x + eA_x/c \\ \Pi_y &= p_y + eA_y/c \end{aligned} \tag{3.2b}$$

The fields are related to the vector potential A_x, A_s, A_y by

$$\begin{aligned} B_x &= \frac{1}{1+x/\rho}\left[\frac{\partial}{\partial s}A_y - \frac{\partial}{\partial y}((1+x/\rho)A_s)\right] \\ B_s &= \frac{\partial}{\partial y}A_x - \frac{\partial}{\partial x}A_y \\ B_y &= \frac{1}{1+x/\rho}\left[\frac{\partial}{\partial x}((1+x/\rho)A_s) - \frac{\partial}{\partial s}A_x\right] \\ \mathcal{E}_s &= -\frac{1}{c}\frac{\partial A_s}{\partial t} \end{aligned} \tag{3.3}$$

It then follows that transfer functions found by integrating Eqs. (3.1) exactly are symplectic transfer functions. The phrase transfer functions is used here to indicate the set of functions that relate the final coordinates to the initial coordinates.

4 SUMMARY

This section summarizes the results found for the transfer functions when longitudinal fields are present. Further details and the derivations of the results are given in reference 4. The results given below might be used in writing a symplectic tracking program when longitudinal fields are present. In the following, $q_x = p_x/p$ and $q_y = p_y/p$, $q_s = \left(1 - q_x^2 - q_y^2\right)^{1/2}$.

For the approximate lattice which is used to generate the integrator, it is assumed that each magnet is broken up into a number of pieces. Each piece is represented in the approximate lattice by point magnets at the ends of the piece and a solenoidal drift between the point magnets. A solenoidal drift is the motion of a particle in a uniform longitudinal field.

The results are given using a reference orbit made up of circular arcs and straight lines which join smoothly. Thus there are regions of the lattice where the reference orbit has a radius of curvalue $1/\rho = 0$, usually at the drifts and quadrupoles, and there are regions where $1/\rho =$ constant, usually at the dipoles. The results can also be used if one chooses a reference orbit which always uses the local cartesian CS, based on the chord that joins the end points of each magnet piece on the reference orbit.

4.1 *Transfer Functions for the Point Magnets*

In the approximate lattice, each magnet piece is represented by point magnets at the ends of the pieces and a solenoidal drift between the ends. The magnet piece goes from $s = s_1$ to $s = s_2$ and has a length along the reference orbit of $h = s_2 - s_1$. For the point magnets, the transfer functions are

$$\begin{aligned} x_2 &= x_1 \quad , \quad y_2 = y_1 \\ q_{x2} &= q_{x1} + \frac{1}{B\rho}\frac{\sin\theta/2}{\theta/2}\frac{h}{2}\hat{B}_y \\ q_{y2} &= q_{y1} - \frac{1}{B\rho}\frac{\sin\theta/2}{\theta/2}\frac{h}{2}\hat{B}_x \\ \theta &= h/\rho = (s_2 - s_1)/\rho \end{aligned} \tag{4.1a}$$

The fields $\hat{B}_x, \hat{B}_y$ may differ from the actual fields at the point magnet by terms of order h and are given by

$$\begin{aligned} \hat{B}_x &= \left[\frac{\partial}{\partial s}A_y(x_1 s\, y_1) - \frac{\partial}{\partial y}((1+x/\rho)A_s(xsy))\right]\frac{1}{1+x/\rho} \\ \hat{B}_y &= \left[\frac{\partial}{\partial x}((1+x/\rho)A_s(xsy)) - \frac{\partial}{\partial s}A_x(x_1 s_1 y_1)\right]\frac{1}{1+x/\rho} \end{aligned} \tag{4.1b}$$

$\hat{B}_x = B_x$, $\hat{B}_y = B_y$ at $s = s_1$ but not at $s = s_2$.

4.2 *Transfer Functions for the Solenoidal Drift*

For the solenoidal drift between the point magnets, the transfer function for x is given by

$$\begin{aligned} x_2 &= x_2^{(0)} + h_1 \\ x_2^{(0)} &= x_1 + \frac{2\rho\sin\theta/2\,(1+x_1/\rho)(q_{x1}\cos\theta/2 + q_{s1}\sin\theta/2)}{-q_{x1}\sin\theta + q_{s1}\cos\theta} \\ h_1 &= \frac{L_{12}q_{\bar{s}1}}{-q_{x1}\sin\theta + q_{s1}\cos\theta}g_1 \\ g_1 &= q_{\bar{x}1}\frac{\sin\alpha - \alpha}{\alpha} + q_{y1}\frac{1-\cos\alpha}{\alpha} \\ \alpha &= -B_{\bar{s}}L_{12}/B\rho \quad , \quad B_{\bar{s}} = B_s(x_1 s_1 y_1)\cos\theta/2 \\ L_{12} &= \frac{1}{q_{\bar{s}1}}\left[1 + (x_1 + x_2)/2\rho\right]2\rho\sin\theta/2 \\ q_{\bar{x}1} &= q_{x1}\cos\theta/2 + q_{s1}\sin\theta/2 \\ q_{\bar{s}1} &= -q_{x1}\sin\theta/2 + q_{s1}\cos\theta/2 \end{aligned} \tag{4.2}$$

$x_2^{(0)}$ is the transfer function when $B_s = 0$. h_1 vanishes when $B_s = 0$, and L_{12} is the path length between s_1 and s_2. h_1 depends on x_2 through L_{12} and α, and Eq. (8.2) is an implicit equation for x_2 which can be solved by iteration, assuming that h_1 can be considered small. This gives the iteration result

$$\begin{aligned} x_2^{(0)} &= x_2^{(0)} \quad , \quad L_{12}^{(0)} = L_{12}\left(x_2^{(0)}\right) \\ x_2^{(1)} &= x_2^{(0)} + h_1\left(L_{12}^{(0)}\right) \quad , \quad L_{12}^{(1)} = L_{12}\left(x_2^{(1)}\right) \\ x_2^{(2)} &= x_2^{(0)} + h_1\left(L_{12}^{(1)}\right) \quad , \quad L_{12}^{(2)} = L_{12}\left(x_2^{(2)}\right) \\ x_2^{(3)} &= x_2^{(0)} + h_1\left(L_{12}^{(2)}\right) \quad , \quad L_{12}^{(3)} = L_{12}\left(x_2^{(3)}\right) \end{aligned} \tag{4.3}$$

Long term tracking is often done with an accuracy of 1 part in 10^{14} in the transfer functions. For large accelerators, where the longitudinal effects are small, the 1 part in 10^{14} accuracy may be achieved after a few iterations.

In doing the iteration indicated by Eq. (4.3), g_1 can be expanded in powers of α, keeping only up to the power of α as the order of the iteration. Thus

$$\begin{aligned}
g_1 &= \sum_{n=1}^{\infty} g_{1n}\alpha^n \\
g_{1n} &= -q_{y1}\frac{(-1)^{\frac{n+1}{2}}}{(n+1)!} \qquad n \text{ odd} \\
g_{1n} &= q_{\overline{x}1}(-1)^{n/2}\frac{1}{(n+1)!} \qquad n \text{ even}
\end{aligned} \tag{4.4}$$

Having found x_2 and L_{12} by solving Eq. (4.2) one can then find q_{x2}, y_2, q_{y2} using

$$\begin{aligned}
q_{x2} &= q_{x1}\cos\theta + q_{s1}\sin\theta + g_2\cos\theta/2 \\
q_{s2} &= -q_{x1}\sin\theta + q_{s1}\cos\theta - g_2\sin\theta/2 \\
y_2 &= y_1 + q_{y1}L_{12} + g_3 \\
q_{y2} &= q_{y1} + g_4 \\
g_2 &= q_{\overline{x}1}(\cos\alpha - 1) + q_{y1}\sin\alpha \\
g_3 &= L_{12}\left[-q_{\overline{x}1}\frac{1-\cos\alpha}{\alpha} + q_{y1}\frac{\sin\alpha - \alpha}{\alpha}\right] \\
g_4 &= -q_{\overline{x}1}\sin\alpha + q_{y1}(\cos\alpha - 1)
\end{aligned} \tag{4.5}$$

4.3 Transfer Functions when $1/\rho = 0$

In this case no iteration is required as L_{12} does not depend on x_2. The transfer functions for the solenoidal drift

$$\begin{aligned}
x_2 &= x_1 + q_{x1}L_{12} + g_1 \\
q_{x2} &= q_{x1} + g_2 \\
y_2 &= y_1 + q_{y1}L_{12} + g_3 \\
q_{y2} &= q_{y1} + g_4 \\
q_{s2} &= q_{s1} \\
\alpha &= -B_s L_{12}/B\rho \ , \quad B_s = B_s(x_1 s_1 y_1) \\
L_{12} &= (s_2 - s_1)/q_{s1}
\end{aligned} \tag{4.6}$$

The transfer functions for the point magnets when $1/\rho = 0$ are given by Eqs. (4.1) if one puts $\sin(\theta/2)/(\theta/2) = 1$.

The problem of tracking symplectically when longitudinal fields are present was treated in Ref. 5 for the case of hard edge fringe fields.

5 REFERENCES

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