Nonlinear Beam Behaviour Study Using Lie Transforms

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Abstract

The particle motion in sextupole fields of a storage ring is studied analytically. The invariants of motion are constructed up to high orders in the perturbation parameters for a harmonic-expanded Hamiltonian using Lie transforms implemented in the REDUCE program. The phase space patterns of the motion are drawn up in resonant and nonresonant cases. A technique for estimating a dynamic aperture (DA) is described. The results are compared with numerical tracking.

1 INTRODUCTION

Many numerical procedures were developed to simulate the nonlinear particle motion in circular accelerators. A bounded area can be estimated in a realistic way taking into account such effects as energy oscillations, closed orbit distortion or the influence of insertion devices. But powerful and complex tracking codes are of the kind of "a black box" for users: sometimes it is difficult to interprete the phenomena and understand either they are resulted from the features of an introduced model or from the code peculiarities.

So the application of analytical methods to get more information out of the tracking and for a better understanding of the underlying physics is of great importance.

In this paper we discuss the results of the single-particle dynamics study for a 2.5 GeV dedicated SRS SIBERIA-2 [1] using the perturbation technique based on Lie transforms. Phase space portraits are shown in 1D (both nonresonant and resonant) and 2D cases including high order effects. The dependence of the horizontal DA on the initial betatron frequency was founded semi-analytically. The estimations are compared with the tracking simulations [2].

2 PHASE SPACE PICTURES

The 2D Hamiltonian with sextupole perturbation in action-angle variables is given by:

$$\begin{aligned} H &= \nu_x J_x + \nu_z J_z \qquad (1) \\ &+ J_x^{3/2}(\sum_{-\infty}^{\infty} A_{1m} cos(\varphi_x - m\theta) \\ &+ \sum_{-\infty}^{\infty} A_{3m} cos(3\varphi_x - m\theta)) \\ &+ J_x^{1/2} J_z(\sum_{-\infty}^{\infty} B_{1m} cos(\varphi_x - m\theta) \\ &+ \sum_{-\infty}^{\infty} B_{-m} cos(\varphi_- - m\theta) \\ &+ \sum_{-\infty}^{\infty} B_{+m} cos(\varphi_+ - m\theta)), \end{aligned}$$

where A_{1m}, A_{3m}, B_{1m} and B_{+-m} are the azimuthal Fourier harmonics of perturbation. With this harmonic-expanded form of Hamiltonian, we can study the influence of selected terms on the particle motion and try to simplify the expressions by leaving only the dominant harmonics.

The unperturbed part of (1) is linear, and nonlinearity appears in the second order of perturbation. It is known that such systems are very sensitive to high orders of perturbation and to construct the solution systematically, order by order, we use the perturbation method of Deprit, which is based on Lie transforms [3]. Deprit recursive relations for the nth order term of the generating function w_n, new Hamiltonian $\bar{H}$ and the transformation operator $\hat{T}$ were implemented in the algebraic manipulation program REDUCE.

First, we consider the horizontal motion. In this case the perturbing part of (1) is

$$H_1 = J_x^{3/2}(\sum_{-\infty}^{\infty} A_{1m} cos(\varphi_x - m\theta) \qquad (2)$$

$$+ \sum_{-\infty}^{\infty} A_{3m} cos(3\varphi_x - m\theta)).$$

Here we assume that the unperturbed betatron tunes are far from resonances which can destroy the convergency of the Deprit series. As we pick the new Hamiltonian $\bar{H}$ to be the nonzero φ_x and θ average terms that produce a secularity in the generating function, the new Hamiltonian $\bar{H}(\bar{J})$ together with the new action $\bar{J}$ are invariants of motion and the inverse transform gives us the relation between the old and new variables $F(\varphi, J) = \bar{J} = const$ whose solution provides the phase trajectories of the original system.

The new action as a function of the old variables can be written ($\theta = 0$ is assumed) to third order:

$$\begin{aligned}
\bar{J} &= \bar{J}_0 + \bar{J}_1 + \bar{J}_2 + \bar{J}_3 + \cdots \qquad (3)\\
\bar{J}_0 &= J_x\\
\bar{J}_1 &= J_x^{3/2}(a_1 cos(\varphi_x) + a_3 cos(3\varphi_x))\\
\bar{J}_2 &= J_x^2(c_0 + c_2 cos(2\varphi_x) + c_4 cos(4\varphi_x))\\
\bar{J}_3 &= J_x^{5/2}(d_1 cos(\varphi) + d_3 cos(3\varphi_x)\\
&+ d_5 cos(5\varphi_x))
\end{aligned}$$

The coefficients in (3) are infinite sums, for example, $c_2 = \sum_{m=-\infty}^{m=\infty} c_{2m}$. The terms of these sums up to the 2nd order are expressed as follows:

$$a_{1m} = \frac{A_{1m}}{\nu_x - m}, a_{3m} = \frac{A_{3m}}{3\nu - m};$$

$$c_{0m} = \sum_{n=-\infty}^{n=\infty} (3a_{1m}a_{1n+m} + a_{3m}a_{3n+m}),$$

$$c_{2m} = \sum_{n=-\infty}^{n=\infty} \frac{2m - n}{2\nu_x - n} a_{1m}a_{3n+m},$$

$$c_{4m} = \sum_{n=-\infty}^{n=\infty} \frac{4m - n}{4\nu_x - n} a_{1m}a_{3n-m},$$

Because of the denominator in a_{1m} and a_{3m} which is the distance from a certain resonance, we can select "strong" harmonics already in a first order expression for $\bar{J}$. In the case of SIBERIA-2:

$$\frac{1}{\nu_x - 1} \approx 3.5, \quad \frac{1}{\nu_x - 2} \approx 1.4$$
$$\frac{1}{3\nu_x - 4} \approx 7.1, \quad \frac{1}{3\nu_x - 3} \approx 1.2.$$

Here ν_x is the betatron tune for one cell, i.e. $\nu_x = Q_x/N_c$. For SIBERIA-2 number of cell $N_c = 6$. The major harmonics a_{11} and a_{34} exceed the next ones approximately by factors of 2 and 6 respectively. As a_{1m} and a_{3m} will apear in high order expressions in products, the influence of all harmonics except the dominant ones is expected to fall off rather quickly.

In Figure 1 the phase trajectories obtained from tracking and predicted i) with the sum over ±300 in (2), ii) with a single harmonic approximation are shown.

The equation for a ith order generating function has the form [3]

$$\hat{D}_0 w_i = \bar{H}_i + F_i(\varphi_x, \theta) \qquad (4)$$

In resonant case we need to choose $\bar{H}_i$ to cancel both the secular terms and the term with a small resonance denominator. Now we cannot transform to a Hamiltonian independent of the phase, but this Hamiltonian can be solved in ordinary way by using the resonant variables. From resonant Hamiltonian which is a function of the new variables $\bar{H}_r(\bar{\varphi}, \bar{J})$ we can find out the dependence $\bar{J}(\bar{\varphi})$ and plot, after back transformation, phase curves of the original problem.

Figure 2 demonstrates the phase space structure in the vicinity of the resonance $5\nu_x = 6$ which appears in 3rd order. The curves are plotted from tracking and resonance perturbation technique up to 5th order. The agreement seems quite reasonable.

In 2D case the phase space calculation using perturbation methods is very cumbersome and can be done under the REDUCE only in a sin-

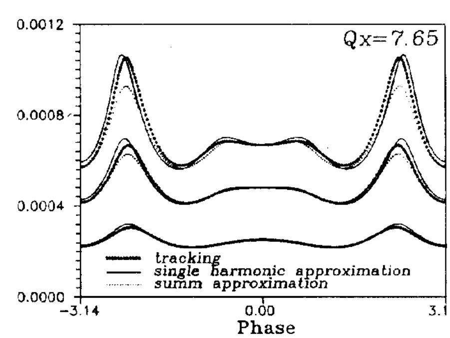

Figure 1: Horizontal phase-space trajectories.

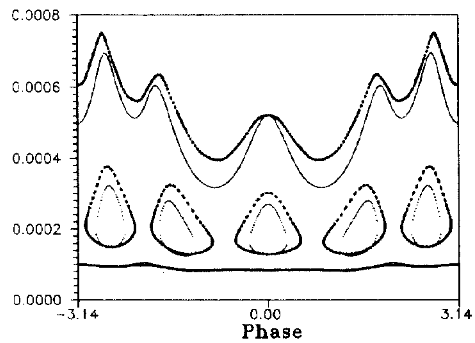

Figure 2: Invariant curves near 3rd order resonance $5\nu_x = 6$.

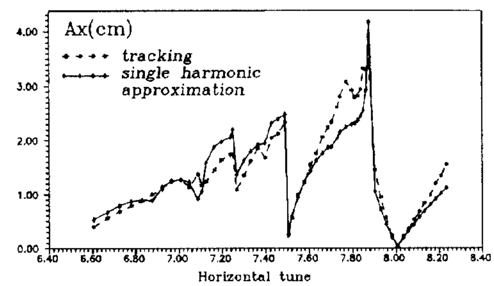

Figure 4: Horizontal DA versus unperturbed betatron tune. Tracking and 5th order prediction.

gle harmonic approximation. In each series in (1) we drop all the terms except main ones. Figure 3 shows the two dimensional 3rd order calculation for phase surfaces $J_z(\varphi_x, \varphi_z)$ in the neighbourhood of a coupling resonance $\nu_x - \nu_z = 0$.

3 DYNAMIC APERTURE

The bounded region is obtained using the expression for a back transformed action variable. We seek the singularity in the $J = f(\varphi)$ which occure when the derivative $\frac{dJ}{d\varphi} = \infty$. It can be shown that last condition is equivalent to $\frac{\partial J(J,\varphi)}{\partial J} = 0$. To find the boundary of stable motion we need to solve last equation numerically and take such a value of φ to which a minimum of J corresponds. This point (J_s, φ_s) when projected to the value of $\varphi = 0$ (Poincare crossection) gives us the DA.

For DA prediction when the betatron tune is irrespective of the resonance points we should make use both resonant and nonresonant perturbation techniques. In Figure 4 the profile of the horizontal DA as a function of the initial betatron tune is demonstrated. The Deprit theory up to 5th order was applied to obtain the expression of action-phase dependence of the original problem. For comparison the numerical simulation results are shown. It seems for us that the values of the stable area boundary for tracking and analytic computation are in good agreement. On the picture one can see resonances $6\nu_x = 42$ (4th order), $5\nu_x = 36$ (3rd order) and $4\nu_x = 30$ (2nd order). The strongest resonance is as one should expect the first order resonance $3\nu_x = 24$.

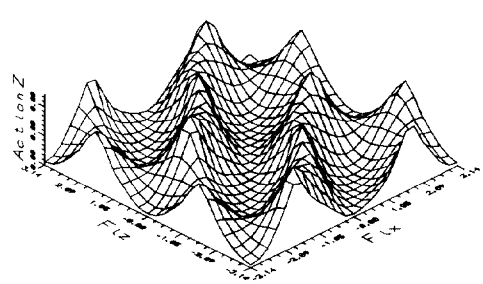

Figure 3: Vertical invariant phase surface.

4 REFERENCES

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