Stochastic Acceleration of Charged Particles by an Electromagnetic Wave in an External Magnetic Field *

V.A.Buts, O.V.Manuilenko, A.P.Tolstoluzhskii
National Science Center
"Kharkov Institute of Physics and Technology"
310108 Academicheskaya Str.1, Kharkov, Ukraine

Abstract

The results of an investigation of the dynamics of charged particles motion in a constant external magnetic field and affected by the regular electromagnetic wave with arbitrary polarization are presented. In this case the stochastic acceleration occurs as a result of the nonlinear cyclotron resonances overlapping. Expressions for the diffusion coefficient in energy space have been revealed and it was demonstrated that its value essentially depends not only on the adjacent (for the particles with given energy γ) cyclotron resonances but on the far ones as well. The conditions under which the width of nonlinear resonances grows faster than the distance between them with an increase of energy have been determined. These conditions are similar to those for unlimited charged particles stochastic acceleration. It was shown that in approaching to the autoresonance conditions there is not stochastic instability at all. Geometrical analysis of wave-particle resonance conditions and integral of motion in energy-momentum space was carried out.

1 General Equations and Results

We consider the motion of a charged particle in a constant, externally applied magnetic field $\vec{H}_0 = \{0, 0, H_0\}$ and in the field of an electromagnetic plane wave:

$$\begin{aligned} \vec{\mathcal{E}} &= Re\{E_0\vec{\alpha}\exp(i\vec{k}\vec{r} - i\omega t)\} \\ \vec{\mathcal{H}} &= Re\{\tfrac{c}{\omega}[\vec{k}\vec{\alpha}]E_0\exp(i\vec{k}\vec{r} - i\omega t)\} \end{aligned} \qquad (1)$$

where $\vec{\alpha} = \{\alpha_x, i\alpha_y, \alpha_z\}$ is the polarization vector of the wave, wave vector $\vec{k}$ has only two components k_x and k_z ($\vec{k} = \{k_x, 0, k_z\}$). In term of the dimensionless variables ($t \to \omega t, \vec{r} \to \vec{r}\omega/c, \vec{p} \to \vec{p}/mc, \vec{k} \to \vec{k}c/\omega$ the equations of particle motion can be reduced to the form

$$\begin{aligned} \dot{\vec{p}} &= (1 - \vec{k}\vec{p}/\gamma)Re(\vec{E}e^{i\psi}) + (\omega_h/\gamma)[\vec{p}\vec{H}] + \\ &(\vec{k}/\gamma)Re[(\vec{p}\vec{E})e^{i\psi}] \\ \dot{\vec{r}} &= \vec{p}/\gamma; \quad \dot{\psi} = \vec{k}\vec{p}/\gamma - 1; \end{aligned} \qquad (2)$$

where $\vec{H} = \vec{\mathcal{H}}/H_0$, $\omega_h = eH_0/mc\omega$, $\vec{E} = e\vec{\mathcal{E}}/mc\omega$, $\psi = \vec{k}\vec{r} - t$, $\vec{k}$ is a unit vector in the direction of wave propagation, $\gamma = (1 + p_z^2 + p_\perp^2)^{1/2}$ is the particle energy, and $\vec{p}$ is its momentum. The set of Eqs.(2) has the integral of motion

$$\vec{p} - Re(i\vec{\mathcal{E}}e^{i\psi}) + \omega_h[\vec{r}, \vec{H}] - \vec{k}\gamma = const \qquad (3)$$

For subsequent analysis, it is convenient to transform to new variables $p_\perp$, p_z, θ, ζ, η defined by

$$\begin{aligned} &p_x = p_\perp\cos\theta,\ p_y = p_\perp\sin\theta,\ p_z = p_z, \\ &x = \zeta - p_\perp/\omega_h\ \sin\theta, y = \eta + p_\perp/\omega_h\ \cos\theta, \end{aligned} \qquad (4)$$

Suppose that the amplitude of field $\vec{\mathcal{E}}_0 = e\vec{E}_0/mc\omega$ is sufficiently small and taken into account that the particle will interact efficiently with the wave if it fulfills one of the resonance conditions:

$$k_z p_z + s\omega_h - \gamma = 0, \quad s = ..., -2, -1, 0, 1, 2, ... \quad (5)$$

after averaging over the fast time scale one can find following equations for particle energy γ and resonant phase $\theta_s = k_z z + k_x\zeta - s\theta - t$ in the case of an isolated resonance

$$\begin{aligned} \dot{\gamma} &= \mathcal{E}_0/\gamma W_s\cos\theta_s, \\ \dot{\theta}_s &= k_z p_z/\gamma + s\omega_h/\gamma - 1 + \mathcal{E}_0 F_s\sin\theta_s, \end{aligned} \qquad (6)$$

where: $W_s = \dfrac{\alpha_x p_\perp s}{\mu}\mathcal{J}_s(\mu) - \alpha_y p_\perp \mathcal{J}_s'(\mu) + \alpha_z p_z \mathcal{J}_s(\mu)$,

$$F_s = s(1 - k_z v_z)(\frac{\alpha_y s}{\mu}\mathcal{J}_s - \alpha_x\mathcal{J}_s') + \frac{k_x^2 v_\perp}{\omega_h}\frac{\alpha_y s}{\mu}\mathcal{J}_s' - \frac{sk_x}{p_\perp}(\frac{\alpha_y p_\perp}{\gamma}\mathcal{J}_s - \alpha_z v_z\mathcal{J}_s') + \frac{k_x\alpha_y}{\omega_h}(k_z v_z - 1)\mathcal{J}_s'$$

$\mu = k_x p_\perp/\omega_h$, $\mathcal{J}_s(x)$ - is a Bessel function of first kind of order s , $\mathcal{J}_s' = d\mathcal{J}_s(x)/dx$. Assuming that

*One of the authors (O.V.M.) acknoledge the IFS Travel Grant Program for financial support.

the $\gamma = \gamma_0 + \tilde{\gamma}_s$, $\tilde{\gamma}_s << \gamma_0$ and taken into account approximate integral of motion $p_z - k_z\gamma = a = const$ one can find the equation of mathematical pendulum, define the width of nonlinear resonance $\Delta\tilde{\gamma}_s$ and the distance between them $\delta\gamma_s$:

$$\begin{aligned} \Delta\tilde{\gamma}_s &= 4\sqrt{\mathcal{E}_0 W_s/k_x^2}, \\ \delta\gamma_s &= \omega_h/k_x^2, \end{aligned} \tag{7}$$

and write generalized Chirikov's criteria [1]:

$$\frac{4\sqrt{\mathcal{E}_0 W_s k_x^2}}{\omega_h} > 1 \tag{8}$$

Under such condition, particles motion becomes complicated, and in fact chaotic.

In the space (γ, p, p_z) particle moves on paraboloid of revolution $\gamma^2 = 1 + p_\perp^2 + p_z^2$. The effective interaction of the particle with the wave is defined by resonance condition which represent cross section of paraboloid by plane $\gamma = k_z p_z + s\omega_h$. We have obtaine ellipses in projection on $(\gamma, p_\perp)$:

$$\frac{(\gamma - s\omega_h/k_x^2)^2}{[(s^2\omega_h^2 - k_x^2)k_z^2/k_x^4]} + \frac{p_\perp^2}{[(s^2\omega_h^2 - k_x^2)/k_x^2]} = 1; \tag{9}$$

and $(p_z, p_\perp)$:

$$\frac{(p_z - s\omega_h k_z/k_x)^2}{[(s^2\omega_h^2 - k_x^2)/k_x^4]} + \frac{p_\perp^2}{[(s^2\omega_h^2 - k_x^2)/k_x^4]} = 1; \tag{10}$$

which at $k_x = 0$ "degenerate" into parabolas:

$$\begin{aligned} \gamma &= (1 + s^2\omega_h^2)/(2s\omega_h) + p_\perp^2/(2s\omega_h);\ (\gamma, p_\perp) \\ p_z &= (1 - s^2\omega_h^2)/(2s\omega_h) + p_\perp^2/(2s\omega_h);\ (p_z, p_\perp) \end{aligned}$$

and at $k_z = 0$ into segment and circle. It is easy to obtain the threshold for value $\tilde{s} = k_x/\omega_h > 1$, at which resonance is possible under fixed wave parameters and $\omega_h < 1$. Integral of motion represented by hyperbolas in projection on $(\gamma, p_\perp)$:

$$\frac{(\gamma - k_z a/k_x^2)^2}{[(k_x^2 + a^2)/k_x^4]} - \frac{p_\perp^2}{[1 + a^2/k_x^2]} = 1; \tag{11}$$

and $(p_z, p_\perp)$:

$$\frac{(p_z - a/k_x^2)^2}{[(k_z^2(1 + a^2/k_x^2)/k_x^2]} - \frac{p_\perp^2}{[1 + a^2/k_x^2]} = 1; \tag{12}$$

which at $k_x = 0$ "degenerate" into parabolas, and at $k_z = 0$ into parabola and straight line. It is easy to see that the integral of motion does not became finite curve. Consequently, it does not restrict gain of energy by particle.

Let us consider interaction of charged particle with a plane circularly polarized electromagnetic wave propagating under angle φ to external magnetic field. In this case W_s can be written as:

$$\begin{aligned} W_s &= \cot(\varphi)s\omega_h\mathcal{J}_s(\frac{\sin\varphi}{\omega_h}p_\perp) \\ &-s\omega_h/\sin(\varphi)\mathcal{J}_s(\frac{\sin\varphi}{\omega_h}p_\perp) \\ &+p_\perp\mathcal{J}_{s+1}(\frac{\sin\varphi}{\omega_h}p_\perp) - \sin(\varphi)p_z\mathcal{J}_s(\frac{\sin\varphi}{\omega_h}p_\perp) \end{aligned} \tag{13}$$

where p_z *and* $p_\perp$ are connected by the relation (10). At $\varphi = 0$ integral coincidence with resonant condition ($s = -1, 1$). This means autoresonance interaction. In this case stochastic instability does not arise. Indeed, from resonance condition and integral of motion one may find following expression for p_z and $p_\perp$:

$$\begin{aligned} p_\perp &= \frac{\omega_h}{\sin\varphi}\sqrt{1 - (a^2 + \sin^2\varphi)/\omega_h^2} \\ p_z &= (\omega_h\cos\varphi + a)/\sin^2\varphi \end{aligned}$$

and evaluated the width of nonlinear resonance (7) at small φ:

$$\Delta\tilde{\gamma}_1 = const\sqrt{\mathcal{E}_0/\sin^3\varphi} \tag{14}$$

We find from (8), (14) that at $\varphi \to 0$
$\Delta\tilde{\gamma}_1/\delta\tilde{\gamma}_1 \to 0$ i.e. according to Chirikov's criteria stochastic instability does not became.

Let us seek out that angle φ, at which with growth energy of the particle the width of the nonlinear resonances is greater then distance between adjacent resonances. This leads to the diffusion of the particle in energy space, and so the unlimited acceleration of particle may be realized. Assuming that the number of resonance $s >> 1$ from (5) and integral for p_z and $p_\perp$ (12) we find:

$$p_z = (s\omega_h\cos\varphi + a)/\sin^2\varphi$$

$$p_\perp = s\omega_h\sqrt{1 - (a^2 + \sin^2\varphi)/(s^2\omega_h^2)}/\sin\varphi$$

Using this relations one can estimate width of nonlinear resonance

$$\Delta\tilde{\gamma}_s = 4\sqrt{\mathcal{E}_0\omega_h c_1 s^{1/3}/\sin^3\varphi}, \quad c_1 \sim 0.411$$

and using Chirikov's criteria and (7) one can obtain condition for the sought angle φ :

$$16\mathcal{E}_0 c_1 s^{1/3}\sin\varphi/\omega_h > 1 \tag{15}$$

As stochastic instability develops, particle trajectories become exceedingly complicated, and are only amenable to study by numerical methods. Sometimes, however, this very complexity makes it possible to simplify the problem considerable by utilizing the methods of statistical physics. Let us consider the evolution of the distribution function for an ensemble of oscillators in a constant magnetic field H_0 and in the field of an arbitrarily polarized external electromagnetic wave. Mutual interaction of particles and wave excitation by particles will be neglected . Under these conditions, the problem of the motion of the ensemble reduces to a one-particle problem. The criterion for the onset of stochastic instability is given for each particle by(8); we shall assume that the electric field strength of the external electric field satisfies this condition. In order to study the diffusion in energy of the particles belonging to the ensemble, we use Eqs.(6) rewritten as

$$\dot{\gamma} = \mathcal{E}_0 \sum \frac{W_n \cos\theta_n}{\gamma}, \dot{\theta}_n = \frac{k_z p_z}{\gamma} + \frac{n\omega_h}{\gamma} - 1 \quad (16)$$

Since stochastic instability will have set in, we may assume that the phase of the resonances are random and independent. Bearing in mind that the terms on the right-hand side of the first of Eqs.(16) are of small amplitude, we may substitute the unperturbed values of the variables. Then the first equation in (16) yields the following expression for the correlation function:

$$\begin{aligned} \mathcal{K} =& < \dot{\gamma}(t+\tau)\dot{\gamma}(t) > = \tfrac{1}{2}(\mathcal{E}_0/\gamma)^2 \\ &\times Re\{exp[i\tau(k_z V_0 - 1)] \sum W_n^2 exp(is\omega_h \tau/\gamma)\}, \end{aligned} \quad (17)$$

where

$$< \gamma > = \int_0^{2\pi} \frac{dk_z}{2\pi} z_0 \frac{1}{2\pi} \int_0^{2\pi} d\phi_0 \gamma \quad (18)$$

Using the addition theorem for cylindrical functions [2], we can expand the sum on the right-hand side of (17) to yield

$$\begin{aligned} \mathcal{K} =& \frac{1}{2}(\frac{\mathcal{E}_0}{\gamma})^2 Re\{e^{i\tau(k_z V_0 - 1)} \\ &\times[(\alpha_+^2 e^{(-i\omega_h \tau/\gamma)} + \alpha_-^2 e^{(i\omega_h \tau/\gamma)})\frac{p_\perp^2}{4}\mathcal{J}_0(\Pi) \\ &+\alpha_z^2 p_\parallel^2 \mathcal{J}_0(\Pi) + \frac{1}{2}p_\perp^2 \alpha_+ \alpha_- \kappa e^{(i\omega_h \tau/\gamma)} \mathcal{J}_2(\Pi) \\ &+p_\perp p_\parallel \alpha_z \mathcal{J}_1(\Pi)\kappa^{1/2}(\alpha_+ + \alpha_- e^{(i\omega_h \tau/\gamma)})]\} \end{aligned} \quad (19)$$

where

$\alpha_\pm = \alpha_x \pm \alpha_y, \kappa \equiv (1 - e^{(-i\omega_h \tau/\gamma)}/(1 + e^{(i\omega_h \tau/\gamma)}),$
$\Pi \equiv 2k_x p_\perp \sin(\omega_h \tau/2\gamma)/\omega_h.$

Making use (19), we can easily find the variance $\sigma^2 \equiv << (\Delta\gamma)^2 >>$. Indeed Eq.(16) gives

$$< \Delta\gamma > = \int_0^t \dot{\gamma}(t')dt' \quad (20)$$

so for the variance we have

$$\sigma^2 = 2\int_0^t d\tau(t-\tau)\mathcal{K}(\tau). \quad (21)$$

At small times

$$t << t_0 << min\left[\frac{\gamma}{\omega_h}, (k_z V_0 - 1)^{-1}, 2^{3/2}\gamma/k_x p_\perp\right]$$

the quantity $\mathcal{K}(\tau)$ can be treated as a constant, yielding a quadratic dependence for the variance:

$$\sigma^2 = \mathcal{K}(0)t^2. \quad (22)$$

At large times, the main contribution to the integral (21) comes from values of τ less then t_0. We then obtain

$$\sigma^2 = \mathcal{K}(0)t_0 t. \quad (23)$$

Note that the value of $\mathcal{K}(\tau)$ and σ^2 essentially depends not only on the neighbourhood cyctlotron resonances but on the far ones as well.

According to the numerical results (see, for example [3], the mechanizm considered here for particle interactions with a field can be an efficient means of heating and accelerating charged particles — the mean particle energy, in time of order 100 cyclotron periods, increased from $< \gamma > = 2$ to $< \gamma > = 5$.

2 References

[1] G.M.Zaslavskii, B.V. Chirikov. "Stochastic Instability of Nonlinear Oscillations". Soviet Physics Uspekhi. vol.105, No.1, p.3, 1971.

[2] I.S.Gradshtein and I.S.Ryzhik,*Tables of Integrals, Sums, Series, and Products*, 5th ed., Nauka, Moscow, (1971) [I.S.Gradshtein and I.S.Ryzhik,*Tables of Integrals, Sums, Series, and Products*, transl. 4th Russian edition. Academic Press, New York (1980)].

[3] V.A.Balakirev, V.A.Buts, A.P.Tolstoluzhskii, Yu.A.Turkin. "Evolution of Distribution Function of Charged Particles Moving in an External Magnetic Field Under Field of Plane Electromagnetic Wave", Ukr. Fiz. Zh., vol.30, No.5, 1985.