A General Control Model for Designing Beam Control Feedback Loops

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Abstract

To control the beam in the synchrotron there may be six different primary feedback loops interacting with the beam at a given time. Three loops are local to the rf cavity. They are: high bandwidth cavity phase and amplitude loops used to minimize the effects due to beam loading and a low bandwidth cavity tuning loop. The loops global to the ring accelerating system are: a radial loop to keep the beam on orbit, a beam phase loop to damp the dipole synchrotron oscillations, and a synchronization loop to essentially lock with the succeeding machine. There are various ways in which these loops may be designed. Designs currently in use in operating machines are based on classical frequency domain techniques. To apply modern feedback controllers and study the interaction of all the feedback loops, a good mathematical model of the beam is extremely useful. In this paper we show the derivation of a non-linear tracking model in terms of differential equations obtained from a set of time varying finite difference equations. The model compares well with the results of thin element tracking codes.

1 INTRODUCTION

Several feedback loops, associated with a basic low-level rf system, have to be able to bunch the beam, accelerate without inducing unwanted coherent oscillations and thereafter time the bunch positions relative to the next higher energy machine for synchronization. For this purpose, a precise control of the frequency phase and amplitude of the accelerating rf signal is required. With a good control model we would benefit a great deal while planning and configuring the feedback loops. There are several ways the beam control loops are designed. One conceivable way is by actually measuring the transfer functions of each parameter from the control end and then designing the appropriate stabilizing dynamics such as the proportional, differential or integral terms. Such an approach has at lease two drawbacks, (i) Existence of some kind of operating machine to conduct experiments and hence to improve the loop performance, (ii) Inaccurate measurements due to difficulties in considering coupling effects between loops. Alternatively, by extracting the model from the longitudinal beam dynamics, we can design the loops more appropriately. Since almost all the practical implementations of the loops are sensitive to errors, a control model with appropriate error terms is even more useful. In this paper we have shown the derivation of a control model by ignoring the local cavity feedback loops, and hence will be applicable to only low intensity machines.

2 PARTICLE TRACKING MODEL

From the control point of view, it will be useful to have the model in differential equation form, although the acceleration takes place only at the cavities. However, a discrete representation will be a starting point since it is closer to reality. Hence a tracking code for one particle was obtained to model the longitudinal phase oscillations by giving an energy kick every time the particle passes through the equivalent cavity gap. This model is shown in Reference [1]. The betatron oscillations are decoupled in the formulation of such models. The error introduced with this type of approximation is very little when there is substantial difference between the betatron and synchrotron frequencies. Since for beam control purpose we are interested in the phase of the particle with respect to the rf signal, it is calculated by knowing the total arrival time of the particle. Using the discrete model we tracked one particle for the Low Energy Booster at $t = 0$ to $0.05\,\text{sec}$ with a Gaussian noise in the magnetic field errors. The results compare quite well with the Thin Element Particle Tracking Code by going through each magnetic lens at 5 ns time steps. Comparisons are shown in Reference [1].

3 NON-LINEAR BEAM CONTROL MODEL

The beam control model is derived below using the discrete model at first for a single particle, and later we show the model for a multiparticle case.

3.1 Synchronization Model

The synchronization loop with "trip-plan" approach [2] provides the means to phase-lock the reference bunch in the lower energy machine with a reference bucket in the higher energy machine. For simplicity let us consider the bunch comprising just one particle. If t_k^i is the time when the reference particle in the lower energy machine reaches the reference point in the kth turn, then the "trip-plan" is

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given by

$$(S_k)_{\text{Trip-plan}} = v_k^s \left(t_k^i - \tilde{t}_k^i\right). \quad (1)$$

The superscript "s" is used to indicate the parameters for synchronous particle. If "trip" is the measured phase for a non-ideal particle, then for kth turn it is given by

$$(S_k)_{\text{actual}} = (v_k^s + \delta v_k)\left(t_k - \tilde{t}_k^i\right) + \delta S_0 \quad (2)$$

where t_k is the actual traversal time of the particle in the lower energy machine. This can be written in the following form for a machine operating below transition:

$$t_k = t_k^i - \sum_{n=1}^{k} \delta\tau_n. \quad (3)$$

The error in synchronizing phase is obtained by subtracting Eq. 1 from Eq. 2. By ignoring the second order terms and converting the discrete error equation to continuous form, the phase error is written as

$$\delta S \cong -v^s \int \frac{\delta\tau}{\tau^s} dt + \delta S_0. \quad (4)$$

The deviation in time, $\delta\tau$, in one traversal can be expressed in terms of radial orbit shift and the field error.

$$\frac{\delta\tau}{\tau^s} = \eta^s \gamma_T^2 \frac{\delta R}{R^s} - \frac{1}{\gamma^2}\frac{\delta B}{B^s} \quad (5)$$

Using Eq. 5 in Eq. 4, the phase error can be expressed in the measurable quantities

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + d_{11}\delta B - a_{11}x_0 \quad (6)$$

where the new variables are shown in Table 1.

3.2 *Radial Orbit Model*

If E_k is the energy in kth pass through the cavity gap, then the energy for actual particle, and a synchronous particle is given by the following difference equations [3]:

$$E_k - E_{k-1} = e(V_k^s + \delta V_k)\sin\phi_k \quad (7)$$

and

$$E_k^s - E_{k-1}^s = eV_k^s \sin(\phi_k^s)_0 \quad (8)$$

where δV_k is given by

$$\delta V_k = \delta V_k^c + \delta V_k^e \quad (9)$$

with δV_k^c as the control supplied to the cavity gap voltage and δV_k^e the error in the cavity voltage for kth turn and $(\phi_k)_0^s$ is the particle phase for the ideal synchronous case. The quantity δV_k^c can be set to zero when we do not use global amplitude feedback. The energy equation is in the finite difference form. It can be transformed to a differential equation in the usual way as follows:

$$\frac{\dot{E}}{v} - \frac{\dot{E}^s}{v^s} = \frac{eV^s}{2\pi R^s}\left[\left(1 + \frac{\delta V}{V^s}\right)\sin(\phi) - \sin\left(\phi_0^s\right)\right] \quad (10)$$

where E and E^s are assumed to be equal to the energy gain per turn of the actual particle and the synchronous particle respectively shown by Eqs. 7 and 8. If δv is the change in velocity from the synchronous particle, then by using Taylor Series approximation, Eq. 10 can be written as below:

$$\frac{d\delta E}{dt} = A_3\left[\left(1 + \frac{\delta V}{V^s}\right)\sin(\phi) - \sin(\phi_0^s)\right] + A_4\delta v \quad (11)$$

where the new variables are shown in Table 1.

The particle phase, ϕ can be written in terms of the nominal synchronous phase, ϕ_0^s, the deviation from the synchronous phase representing the synchrotron oscillations, $\delta\phi^s$, the systematic phase error, ϕ^e, and also, a small phase shift, $\delta\phi^c$ as supplied by the controller. That is,

$$\phi = \phi_0^s + \delta\phi^c + \delta\phi^s + \phi^e. \quad (12)$$

The phase shift $\delta\phi^c$ is included as one of the control inputs, since the radial loop can be connected to the global phase shifter after the frequency source, as in the case of Fermilab booster low level rf system. We can write the following functional relationship between energy and momentum

$$\delta E = (\beta^s)^2 \frac{\delta P}{P^s} E^s. \quad (13)$$

Substituting the well known equation for the momentum change from Reference [1], and by taking the first derivative of the resulting energy equation with respect to time we obtain:

$$\frac{d\delta E}{dt} = \dot{A}_1\delta R + A_1\frac{d\delta R}{dt} + A_2\frac{d\delta B}{dt} + \dot{A}_2\delta B. \quad (14)$$

The incremental velocity change in a given turn has a functional relationship:

$$\frac{\delta v}{v^s} = \frac{1}{(\gamma^s)^2}\left[\gamma_T^2\frac{\delta R}{R^s} + \frac{\delta B}{B^s}\right].$$

It is substituted in Eq. 14 and then the resulting equation is compared with Eq. 11. After simplification we get the desired equation for the radial orbital deviations as follows:

$$\begin{aligned}\dot{x}_2 \;=\; & a_{22}x_2 + (a_{23} - \tilde{a}_{23}\omega)\sin(x_3 + x_4) \\ & + (a_{24} - \tilde{a}_{24}\omega)\cos(x_3 + x_4) \\ & + \tilde{a}_{24} + d_{21}\delta B + \tilde{d}_{21}\delta\dot{B} \end{aligned} \quad (15)$$

where the new variables are shown in Table 1.

3.3 *Particle Phase Model*

The discrete phase equation is well known and is written below with error and control terms

$$\phi_{k+1} = \phi_k + 2\pi(f_k^s + \delta f_k^s + \delta f_k^c)(\tau_k^s + \delta\tau_k) + \phi^e. \quad (16)$$

By substituting the equation for $\delta\tau_k$ in terms of the radial orbit shift and the magnetic field errors and converting the

Table 1: Parameters of the state space model.

Coefficients:	
$a_{11} = \frac{\dot{v}^s}{v^s}$	$d_{11} = \frac{v^s}{(\gamma^s)^2 B^s}$
$a_{12} = -\frac{v^s \eta^s \gamma_T^2}{R^s}$	$d_{21} = \frac{A_5 - \dot{A}_2}{A_1}$
$a_{22} = \frac{A_4 - \dot{A}_1}{A_1}$	$\tilde{d}_{21} = -\frac{A_2}{A_1}$
$a_{23} = \frac{A_3}{A_1}\left(1 + \frac{\delta V^c}{V^s}\right)\cos\phi_0^s$	$d_{31} = \frac{2\pi f^s}{(\gamma^s)^2 B^s}$
$\tilde{a}_{23} = -\frac{A_3}{A_1}\cos\phi_0^s$	$d_{32} = \frac{\beta^s c}{2\pi R^s}$
$a_{24} = \frac{A_3}{A_1}\left(1 + \frac{\delta V^c}{V^s}\right)\sin\phi_0^s$	$A_1 = \frac{(\beta^s)^2 \gamma_T^2 E^s}{R^s}$
$\tilde{a}_{24} = -\frac{A_3}{A_1}\sin\phi_0^s$	$A_2 = \frac{(\beta^s)^2 E^s}{B^s}$
$a_{32} = \frac{2\pi f^s \eta^s \gamma_T^2}{R^s}$	$A_3 = \frac{eV^s \beta^s c}{2\pi R^s}$
$b_{31} = 2\pi$	$A_4 = A_3 \frac{\gamma_T^2}{R^s \gamma^2}\sin\phi_0^s$
	$A_5 = A_3 \frac{1}{B^s \gamma^2}\sin\phi_0^s$
Variables:	
$x_1 = \delta S \quad x_2 = \delta R \quad x_3 = \delta\phi^s \quad x_4 = \delta\phi^c$	
$u = \delta f^c \quad \omega = \frac{\delta V^c}{V^s}$	
Errors:	
$\delta B \quad \delta f^e \quad \delta V^e \quad \phi^e$	

equation to continuous form the following state equation is obtained:

$$
\begin{aligned}
\dot{x}_3 &= a_{32}x_2 + b_{31}u + v - \dot{\phi}_0^s + (d_{31}\delta B + b_{31}\delta f^e + d_{32}\phi^e) \\
\dot{x}_4 &= -v
\end{aligned} \tag{17}
$$

where the new notations are shown in Table 1 above. v shown above is not the same as velocity used in Eq. 10.

4 LINEAR STATE SPACE MODEL

The model shown in the previous section can be linearized to a time-varying state space model for small angle phase oscillations. It is given by

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ 0 & a_{22} & a_{23} & a_{23} \\ 0 & a_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\tilde{a}_{24} & 0 & 0 \\ 0 & 0 & b_{31} & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ \omega \\ u \\ v \end{bmatrix} \tag{18}
$$

Clearly, the above equation can be written in a more general state space form as follows:

$$
\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u} \tag{19}
$$

where $\underline{x}$ represents the state vector, $\underline{A}$ represents the system matrix, $\underline{B}$ represents the input matrix and $\underline{u}$ represents the control vector. Eq. 19 is known as state differential equation. With Eq. 18 several linear control combinations can be analyzed and a suitable feedback compensation can be included. The non-linear dynamical equations represented by Eq. 6, 15 and 17 can be used to design loops for large phase angle variation. However, in such cases, the single particle non-linear model will not be very accurate. Hence we show a slight modification for the multiparticle case below.

5 MULTIPARTICLE STATE SPACE MODEL

Let us assume that a bunch-to-bucket transfer is used at injection into the accelerator with a bunch of N particles having energy and phase spread. If $\delta\tau_1, \delta\tau_2, \ldots, \delta\tau_N$ are the traversal times for each particle, then the traversal times can be written in terms of $\delta R_1, \delta R_2, \ldots, \delta R_N$ from Eq. 5. After substituting the orbit shifts for each particle, on the average Eq. 6 can be written as,

$$
\dot{\bar{x}}_1 = a_{11}\bar{x}_1 + a_{12}\bar{x}_2 \tag{20}
$$

where

$$
\bar{x}_1 = \frac{1}{N}\sum_{j=1}^{N}(S_{\text{actual}} - S^j_{\text{Trip-plan}}) \quad \text{and} \quad \bar{x}_2 = \frac{1}{N}\sum_{j=1}^{N}\delta R^j
$$

and error terms do not change. Similarly, Eq. 15 becomes equal to

$$
\begin{aligned}
\dot{\bar{x}}_2 &= a_{22}\bar{x}_2 + (a_{23} - \tilde{a}_{23}\omega)\sin(\bar{x}'_3 + x_4) \\
&\quad + \tilde{a}_{24} + (a_{24} - \tilde{a}_{24}\omega)\cos(\bar{x}'_3 + x_4)
\end{aligned} \tag{21}
$$

with

$$
\bar{x}'_3 = (\delta\bar{\phi}^s) = \text{atan}\left[\frac{\sum_{j=1}^{N}\sin(\delta\phi^s)^j}{\sum_{j=1}^{N}\cos(\delta\phi^s)^j}\right]
$$

$\bar{x}_3$ is defined in Eq. 22 below by rewriting Eq. 17 for multiparticle case:

$$
\dot{\bar{x}}_3 = a_{32}\bar{x}_2 + b_{31}u + v \quad \text{and} \quad \dot{x}_4 = -v \tag{22}
$$

with

$$
\bar{x}_3 = \frac{1}{N}\sum_{j=1}^{N}(\delta\phi^s)^j
$$

For a linear model, clearly $\bar{x}'_3 = \bar{x}_3$ and matrices $\underline{A}$ and $\underline{B}$ do not change.

6 CONCLUSIONS

A general control model is derived in state space form for planning and studying the beam control feedback loops. The model is obtained at first for a single particle case and then it is extended to include multi-particle. Validation tests were carried out with a particle tracking code.

7 REFERENCES

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