Analytic Approach to Design of High γ_t Lattice — 'Missing Magnet' Scheme

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Abstract

Analytic formulas for the matched dispersion η and the transition energy γ_t are derived for a missing magnet lattice. As an example, three lattices having different number or/and location of empty half-cells are considered. Using the numerical program MAD along with derived formulas, the variation of γ_t, peak dispersion $|\eta|_{max}$ with betatron tune are computed in these lattices. Comparison shows a good correspondence between values obtained by using the analytic formulas and those given by MAD.

1 INTRODUCTION

Approximate analytic expressions for lattice functions are useful tool for lattice design, e.g. for a quick estimate of lattice parameters, for guiding numerical minimization programs, e.g. MAD [1]. In early lectures on synchrotron theory [2], analytic formulas of beta functions and dispersion were derived for regular FODO cells. Recently, more general and some new formulas have been derived for regular arrays of FODO and FDO cells [4,5]. More complicated but interesting subject for analytic study is superperiodic lattices, i.e. lattices having a supercell's structure. It is well known, introducing a superperiodicity assume either a modulation of the beta function or a perturbation of the bending radius ρ, or both (see e.g. [3]). The variation of the ρ can be achieved, for example, in a lattice having structure of simple focusing cells, e.g. FODO cells, some of which are empty and others filled with dipole magnets — missing magnet lattice. Missing cells is used not only for design of transitionless lattices, but also for meeting a requirement of providing lattice with free spaces for injection, extraction, acceleration, etc. hardware.

In recent report [4] an analytic approach was proposed to design of transitionless lattices having modulated beta function. Now, going on with analytic study, we intend to derive analytic formulas for lattice functions in missing magnet lattices.

2 THEORY

Let us consider a missing magnet lattice with S superperiods. Let each superperiod has N FODO cells, N_e of which are empty and others are filled with dipole magnets having bending radius ρ. Let us supposed, for simplicity, each superperiod has a mirror symmetry. The empty half-cells we will mark by its ordinal number in a superperiod: j_k , $k = 1, 2 \ldots N_e$, e.g. $j_1 = 1, j_2 = 3$ means that first, third, $2N^{\text{th}}$ and $[(2N+1)-3]^{\text{th}}$ half-cells are empty.

Further let us suppose that dipole magnets are relatively weak and we may neglect their focusing. Then, obviously, beta functions, chromaticity, betatron tunes are almost the same as in the regular lattice (unperturbed) having N cells filled with dipoles. And, therefore, the analytic formulas for these parameters, derived for regular FODO cells in approximation of thin lens and weak focusing of the magnets [2,4,5], are valid for a missing magnet lattice. In this approximation the only effect of empty cells is changes in the dispersion function and transition energy. The matched dispersion η and transition energy γ_t can be represented in terms of Fourier components [6,3]:

$$\eta(\phi) = \sqrt{\beta}\,\nu^2 \sum_{k=-\infty}^{\infty} \frac{a_k}{\nu^2 - k^2}\, e^{ik\phi}\,, \tag{1}$$

$$\gamma_t^{-2} = \frac{Q\,\nu^2}{R} \sum_{n=-\infty}^{+\infty} \frac{|a_n|^2}{\nu^2 - n^2}\,, \tag{2}$$

where harmonic components a_n, generated by missing magnets, are

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} \frac{\beta_x^{3/2}}{\rho} \cos n\phi\, d\phi\,, \tag{3}$$

where $\nu = \mu_s/2\pi$, μ_s is the phase advance over each superperiod in horizontal plane, $Q = \nu S$ the betatron tune of the whole lattice, $\phi(s) = \mu(s)/\nu$ the normalized phase advance, R is the average radius. The eq. (3) is easily found to be

$$a_n = \frac{1}{2\pi}\left(\frac{N}{N-N_e}\right)\Big[\int_0^{2\pi} \frac{\beta_x^{3/2}}{\rho_0} \cos n\phi\, d\phi - \int_{empty} \frac{\beta_x^{3/2}}{\rho_0} \cos n\phi\, d\phi\Big]\,, \tag{4}$$

where ρ_0 is the bending radius in dipole magnets of the unperturbed lattice, and ρ by $\rho_0 = \rho\, N/(N-N_e)$. The second term in (4) is an integral over empty half-cells.

Further, assuming that

$$\frac{1}{2\pi}\int_a^b \frac{\beta_x^{3/2}}{\rho_0} \cos n\phi\, d\phi \approx \frac{\lambda}{R}\,\frac{1}{2\pi n} \sin n\phi\Big|_a^b\,, \tag{5}$$

where $\lambda = <\beta_x^{3/2}>_\phi$, we have from eq. (4)

$$a_n \approx \frac{\lambda}{R}\left(\frac{N}{N-N_e}\right)\left[\delta_{n0} - \frac{2\,\sigma_n \sin\frac{\pi n}{2N}}{\pi n}\right]\,,$$

$$\sigma_n = \sum_{l=1}^{N_e} \cos\left(\frac{\pi n}{2N}(2j_k - 1)\right)\,, \tag{6}$$

where $\delta_{n0} = 1$, if $n = 0$, and $\delta_{n0} = 0$, if $n \neq 0$. Using the Fourier series (1), (2) end eq. (6) we obtain

$$\eta \approx \eta_0 \left[1 - \frac{2\nu^2}{\pi} \left(\frac{N}{N - N_e} \right) \sum_{n=1}^{\infty} \frac{2\sigma_n \sin \frac{\pi n}{2N} \cos n\phi}{n(\nu^2 - n^2)} \right], \quad (7)$$

$$\gamma_t^{-2} \approx \gamma_{t0}^{-2} \left[1 + \frac{2\nu^2}{\pi^2} \left(\frac{N}{N - N_e} \right)^2 \sum_{n=1}^{\infty} \frac{4\sigma_n^2 \sin^2 \frac{\pi n}{2N}}{n^2(\nu^2 - n^2)} \right], \quad (8)$$

where we used the fact that $\gamma_{t0}^{-2} \approx \frac{\lambda^2 Q}{R^3}$, $\eta_0 \approx \beta^{1/2} \frac{\lambda}{R}$, and γ_{t0}, η_0 are the unperturbed transition energy and dispersion.

It is then straightforward, but cumbersome, to evaluate the sums over n in eq. (7), (8), by using formulas taken from ref. [7]. We shall not do this explicitly here, just give the result:

$$\eta(\phi) \approx \eta_0(\phi) \cdot \left[1 - \left(\frac{N}{N - N_e} \right) \sum_{k=1}^{N_e} f_k(\phi) \right], \quad (9)$$

where

$$f_k(\phi) = \begin{cases} \psi_k^1, & \text{if } 0 \le \phi \le \frac{\pi}{N}(j_k - 1); \\ \psi_k^2, & \text{if } \frac{\pi}{N}(j_k - 1) \le \phi \le \frac{\pi}{N} j_k; \\ \psi_k^3, & \text{if } \frac{\pi}{N} j_k \le \phi \le \pi, \end{cases} \quad (10)$$

$$\psi_k^1 = -\frac{1}{N} + \frac{2\cos \pi\phi}{\sin \pi\nu} \sin \frac{\pi\nu}{2N} \cos\left(\frac{\pi\nu}{2N}(2N - 2j_k + 1)\right), \quad (11)$$

$$\psi_k^2 = \frac{(N-1)}{N} - \frac{1}{\sin \pi\nu} \left(\cos \nu\phi \sin\left(\pi\nu \frac{N - j_k}{N}\right) + \sin\left(\frac{\pi\nu}{N}(j_k - 1)\right) \cos(\nu(\pi - \phi)) \right), \quad (12)$$

$$\psi_k^3 = -\frac{1}{N} + \frac{2\cos(\nu(\pi - \phi))}{\sin \pi\nu} \sin \frac{\pi\nu}{2N} \cos\left(\frac{\pi\nu}{2N}(2j_k - 1)\right). \quad (13)$$

And for γ_t we obtain

$$\gamma_t^{-2} \approx \gamma_{t0}^{-2} \left[1 + \frac{N_e}{N - N_e} \left(1 - \frac{2\sigma N^2}{(N - N_e)} \times \frac{\sin(\frac{\pi\nu}{2N}(2N-1)) \sin \frac{\pi\nu}{2N}}{\pi\nu \sin \pi\nu} \right) \right], \quad (14)$$

where

$$\sigma = 1 - \frac{\sin \frac{\pi\nu}{2N}}{N_e \sin(\frac{\pi\nu}{2N}(2N-1))} \times \left[\sum_{k=1}^{N_e} \cos\left(\frac{\pi\nu}{N}(N + 1 - 2j_k)\right) + 4 \sum_{k=1}^{N_e - 1} \sum_{m=k+1}^{N_e} \cos\left(\frac{\pi\nu}{2N}(2N + 1 - 2j_m)\right) \times \cos\left(\frac{\pi\nu}{2N}(2j_k - 1)\right) \right] \quad (15)$$

As it could be expected, η and γ_t depend on not only number of empty cells, but also on its location within a superperiod. These partly cumbersome expressions (9)–(15) become slightly simpler, if the empty half-cells run from 1 to N_e, i.e. $j_k = k = 1, 2, \ldots N_e$. In this case, by evaluating the sums over j_k, we have

$$\eta(\phi) \approx \eta_0(\phi) \cdot \left[1 - \frac{1}{\sin \pi\nu} \left(\frac{1}{N - N_e} \right) \times \begin{cases} \Lambda_1, & 0 \le \phi \le \frac{\pi N_e}{N}; \\ \Lambda_2, & \frac{\pi N_e}{N} \le \phi \le \pi, \end{cases} \right], \quad (16)$$

where

$$\Lambda_1 = (N - N_e) \sin \pi\nu - N \cos \nu\phi \sin\left(\frac{\nu\pi}{N}(N - N_e)\right),$$

$$\Lambda_2 = -N_e \sin \pi\nu + N \sin \frac{\pi\nu N_e}{N} \cos(\nu(\pi - \phi)), \quad (17)$$

$$\gamma_t^{-2} \approx \gamma_{t0}^{-2} \left(\frac{N}{N - N_e} \right) \left[1 - \left(\frac{N_e}{N} \right) \times \frac{sc(\frac{\pi\nu N_e}{N})\, sc(\pi\nu \frac{N - N_e}{N})}{sc(\pi\nu)} \right], \quad (18)$$

where $sc(z) \equiv \frac{\sin z}{z}$.

3 AN EXAMPLE AND COMPARISONS WITH MAD

As an example we consider three missing magnet lattices for 8 GeV synchrotron. All the lattices have eight superperiods, with four FODO cells in each superperiod, but different location of empty half-cells and/or its number. The maximum pole-tip field in dipoles is taken 1 T, gaps between dipoles and quadrupoles are 0.5 m, length of quadrupoles is 1 m. Some other parameters are shown in Table 1. For these lattices we have computed the variation of γ_t and the peak dispersion $|\eta|_{max}$ with betatron tune, by using the numerical program MAD [1] and the formulas (9)–(15). For unperturbed values of dispersion and transition energy η_0, γ_{t0} in (9)–(15) we used its analytic expressions given in [4,5]. The results of comparison between analytic and numerical calculations are shown on the Fig. 1—4.

Table 1: Parameters of missing magnet superperiods for 8 GeV synchrotron.

	Sup. I	Sup. II	Sup. III
Length/sup [m]	47.05	62.57	62.57
N	4	4	4
N_e	1	2	2
j_k	1	1,2	1,3
L_{dipole} [m]	3.88	5.82	5.82

It is seen that there is a good correspondence between values predicted by eq. (9)–(15) and numerical ones. Besides, as it is seen from Fig. 4, the eq. (9) gives not only a good prediction of the peak dispersion but also correctly describes behavior of the dispersion function in a superperiod.

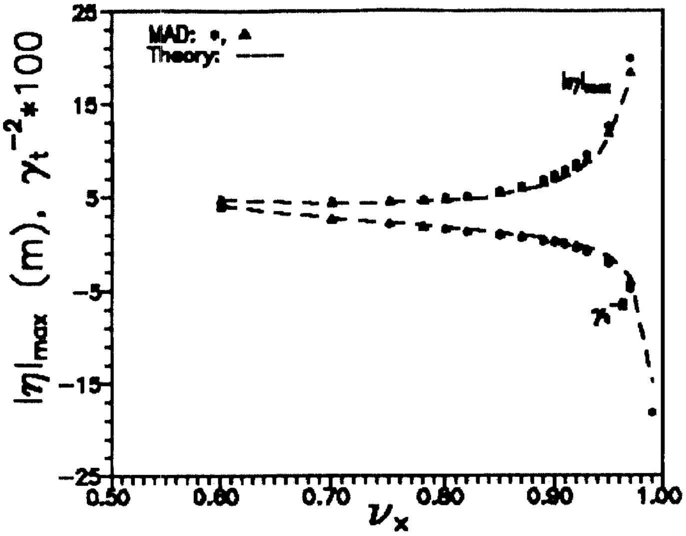

Figure 1: Peak dispersion and γ_t versus tune in missing magnet lattice Sup. I

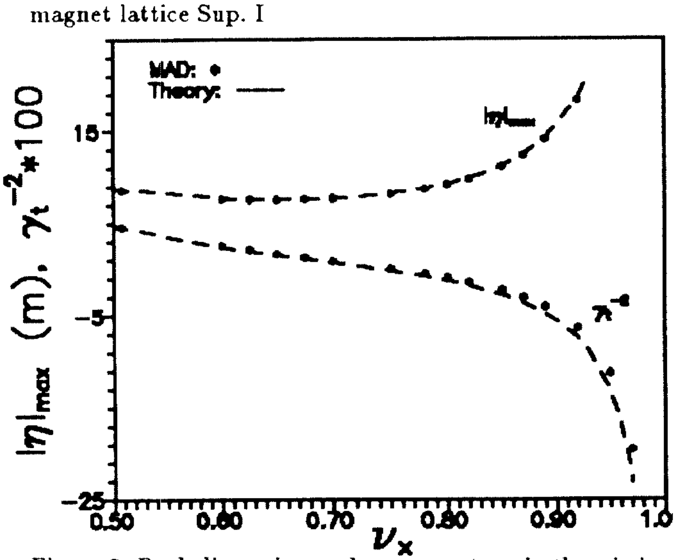

Figure 2: Peak dispersion and γ_t versus tune in the missing magnet lattice Sup. II

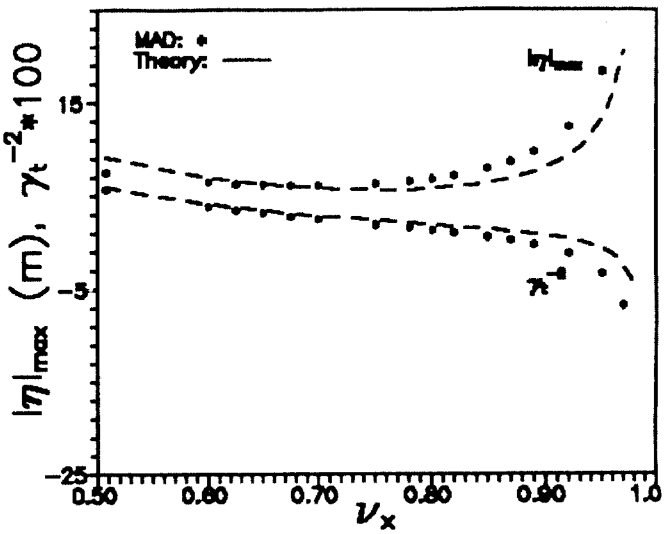

Figure 3: Peak dispersion and γ_t versus tune in the missing magnet lattice Sup. III

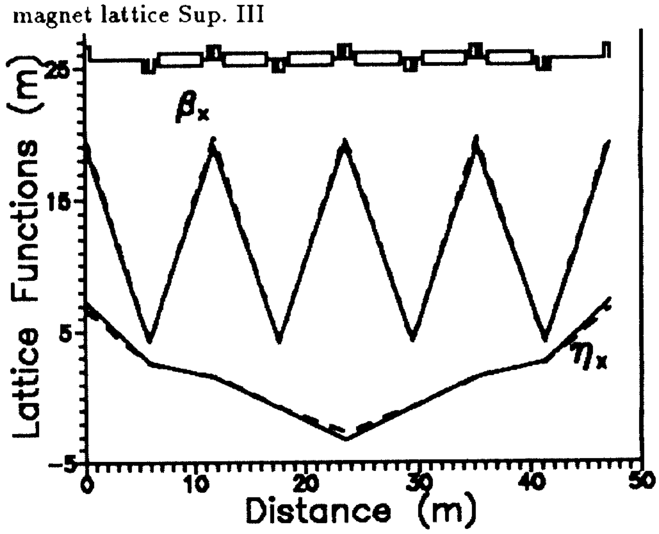

Figure 4: Lattice functions for the missing magnet lattice Sup. I

4 CONCLUSION

The analytic formulas for dispersion and transition energy have been derived in a missing magnet FODO lattice. These formulas, along with formulas for lattice parameters in regular cells, seem to be powerful tool for lattice design.

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6 REFERENCES

[1] F. C. Iselin, J. Nieder, "User's Reference Manual," CERN/SL/90-13 (AP), 1990

[2] E. Keil, "Single-Particle Dynamics," in proc. CERN 77-13, 1977, pp. 11-36

[3] R. C. Gupta, J. I. M. Botman and M. K. Craddock, "High Transition Energy Magnet Lattices," IEEE Trans., Vol. NS-32, pp. 2308-2310, October 1985

[4] A. I. Iliev, "Analytic Approach to Design of High Transition Energy Lattice with Modulated β-function," to be published in proc. of 14th Biennial IEEE Particle Accel. Conf., May 6-9, 1991, San Francisco, CA

[5] A. Iliev, " Analytic Formulas for Lattice Functions for Simple Arrays of Cells," in prepare to be published

[6] E. D. Courant, H. S. Snyder, "Theory of the alternating-gradient synchrotron," Annals of Phys., vol. 3, pp. 1-48, 1958

[7] L. P. Prudnikov, Yu. A. Brychkov, O. I. Marichev, Integrals and Sums, in russian, Moscow, Science, 1981