Nonlinear Beam Dynamics Close to Resonances Excited by Sextupole Fields

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Abstract

The particle motion in a circular accelerator structure with sextupoles close to 4-th and 6th order resonances is studied with perturbation theory. It is shown that a refined method is needed to achieve good agreement between perturbation theory and numerical simulation.

1 Introduction

The understanding of single particle dynamics in the presence of nonlinear forces is an important issue for future circular accelerators. Besides numerical simulations, the need for theoretical models is generally acknowledged. In particular it is necessary to develop analytical tools which can be used together with numerical simulations in order to determine single particle stability. One approach is the concept of isolated nonlinear resonances [1,2,3]. A more recent analytical tool is the concept of the Lyapunov exponent [4] which, being determined by numerical simulation, allows early detection of instability. Another useful criterion for instability is the overlap of stabilized resonance islands as proposed by Chirikov [5].

Operating close to higher order resonances, perturbation theory does not give a good estimate for the onset of instability nor for the width of islands even in simple systems. The aim of this report is to present a method which considerably improves the results of perturbation theory. This can be demonstrated, for example, in the case of the most important nonlinearities in accelerators, the sextupole fields. The reason why sextupole fields are so delicate is that the average value of the nonlinear potential vanishes so that there is no first order contribution to the amplitude dependent tune. The proposed method overcomes the problems related to this fact.

For the sake of clarity, we present only the case of horizontal betatron motion, and we have chosen a simple system, namely a linear lattice with just one localized (thin lens) sextupole. But we want to emphasize that there is no restriction in applying the method for more complex cases.

2 Perturbation Theory

We start from standard perturbation theory [1,2] by considering a nonlinear contribution to the Hamiltonian for a particle travelling around the accelerator:

$$H = \frac{1}{2}x'^2 + \frac{1}{2}kx^2 + \sum_n \frac{b_n}{n}x^n = H_0 + H_1 \tag{1}$$

Inserting the solution for linear motion

$$x(s) = \sqrt{2I\beta(s)}cos(\phi(s) - \phi(0) + \Phi) \tag{2}$$

(where $\beta(s), \phi(s)$ are β-function and betatron-phase advance) in the complete equation of motion, one obtains a canonical system of differential equations for the Courant-Snyder constants I, Φ with H_1 as a new Hamiltonian function. (The subscript of H_1 will be subsequently omitted). The new Hamiltonian will be represented by a Fourier series (index q):

$$H = \sum_{nmq} h_{nmq} I^{n/2} e^{i(m\Phi+(mQ_0+q)\theta)} \tag{3}$$

$$h_{nmq} = \frac{1}{2\pi}\oint ds \left(\begin{array}{c} n \\ \frac{n-m}{2}\end{array}\right)\left(\frac{\beta}{2}\right)^{n/2}\frac{b_n}{n}e^{i(m\phi(s)-(mQ_0+q)\theta)}$$

(where $m = n, n-2, ...2-n, -n$; $\theta = 2\pi s/L$; L=Circumference; Q_0= linear machine tune). Following the principle of perturbation theory, we look for a canonical transformation generated by S

$$H, I, \Phi \rightarrow K, J, \psi \tag{4}$$

such that the new system Hamiltonian, K, depends only on the action variable J where J is a constant of motion

$$J' = \frac{\partial K}{\partial \psi} = 0 \rightarrow J = const. \tag{5}$$

We assume that S and K have the form

$$S = J\Phi + \sum_{nmq}\sigma_{nmq}J^{n/2}e^{i(m\Phi+(mQ_0+q)\theta)} \tag{6}$$

$$K = \sum_{nmq} k_{nmq} I^{n/2} e^{i(m\psi+(mQ_0+q)\theta)}.$$

S, K and H are related by the Hamilton-Jacobi equation

$$K - H = \partial S/\partial s. \tag{7}$$

In order to express this equation with one set of variables J, Φ, we expand the terms $I^{n/2}$ and $e^{im\psi}$ in H and K respectively in a Taylor series in terms of J and Φ, using the relationship:

$$I = \partial S/\partial \Phi, \quad \psi = \partial S/\partial J. \tag{8}$$

Ordering of the terms with the same exponent and making use of linear independence leads to a relationship between the coefficients h_α, k_α (α will subsequently represent n, m, q):

$$i(mQ_0+q)\sum_n \sigma_\alpha J^{n/2} = \sum_n (k_\alpha - h_\alpha)J^{n/2} \tag{9}$$

$$-\sum_{\alpha',\alpha''}\frac{n''m'}{2i}k'_\alpha\sigma''_\alpha J^{\frac{n'+n''-2}{2}} + \sum_{\alpha',\alpha''}\frac{n'm''}{2i}h_{\alpha'}\sigma_{\alpha''}J^{\frac{n'+n''-2}{2}} - \ldots$$

(where $m - m'' = m'$, $q - q'' = q'$). Requiring that the new Hamiltonian K has only terms with $m = q = 0$ or $mQ_0 + q \simeq 0$, (otherwise setting the coefficients $k_\alpha = 0$) and solving the remaining equation for the σ_α by iteration corresponds to the Poincaré-Zeipel [6] procedure as introduced into accelerator physics by Schoch[2], resulting in:

$$\sum_n \sigma_\alpha J^{n/2} = -\sum_n \frac{h_\alpha J^{n/2}}{i(mQ_0+q)} \tag{10}$$

$$+\sum_{\alpha',\alpha''}\frac{n'm''}{2}\frac{h_{\alpha'}h_{\alpha''}J^{\frac{n'+n''-2}{2}}}{i(m''Q_0+q'')(mQ_0+q)} + \ldots$$

and, in the case of $mQ_0 + q \simeq 0$ or $m = q = 0$, σ_α is set to zero, the Hamiltonian coefficients are:

$$\sum_n k_\alpha J^{n/2} = \sum_n h_\alpha J^{n/2} - \sum_{\alpha',\alpha''} \frac{n'm''}{2} \frac{h_{\alpha'} h_{\alpha''} J^{(n'+n''-2)/2}}{(m''Q_0 + q'')} + \dots \quad (11)$$

This procedure is known to converge very slowly. A considerable improvement is achieved by removing any h_{n00} terms from the second order sum eqn(9) if they appear. They can be added to the left hand side of eqn (9).

$$i\sum_n (m(Q_0 + \sum_{n'} \frac{n'}{2} h_{n'00} J^{\frac{n'-2}{2}}) + q)\sigma_\alpha J^{n/2} = \dots \quad (12)$$

Proceeding in this way, the resonance denominators in the expression for the generating function contain, in addition to the linear tune, an action-dependent correction which we identify as the contribution to the detuning terms in the new Hamiltonian K

$$\Delta Q = \frac{1}{2\pi}\int_0^{2\pi} d\theta \partial K/\partial J = \sum_{n'} \frac{n'}{2} h_{n'00} J^{\frac{n'-2}{2}}. \quad (13)$$

In case of odd-order multipole fields (sextupole, decapole,...) however, coefficients h_{n00} do not exist and contributions to the detuning appear only in 2nd or higher order terms according to eqn (11). Therefore it is necessary to consider 3rd order terms of the Taylor expansion eqn(9):

$$\sum_{\alpha',\alpha'',\alpha'''} \frac{n'(n'-2)m''m'''}{8} h_{\alpha'}\sigma_{\alpha''}\sigma_{\alpha'''} J^{(n'+n''+n'''-4)/2} \quad (14)$$

(where m=m'+m''+m''', q=q'+q''+q'''). Separating the terms with $m''' = m$, $q''' = q$ (thus $m' + m'' = 0$ and $q' + q'' = 0$) from the triple sum, replaces the left hand side of eqn (9) by

$$i(m(Q_0 - \sum_{\alpha'\alpha''} \frac{n'(n'-2)m''}{2i} h_{\alpha'}\sigma_{\alpha''} J^{\frac{n'+n''-4}{2}}) + q)\sum_n \sigma_\alpha J^{n/2}. \quad (15)$$

The second order terms which now appear in the resonance denominator resemble [1] the detuning terms obtained in the second order terms of the Hamiltonian K. Equations (9,15) define a continuous fraction series for the coefficient σ_α for each order of the Taylor expansion:

$$\sigma_\alpha = \frac{-h_\alpha}{i(mQ + \sum c' h_{\alpha'} h_{\alpha''}/2(m''Q + \sum c'' h_{\alpha'''} h_{\alpha''''}/2(m''''Q + \dots} + \dots \quad (16)$$

At this point, we propose to replace the complicated continuous fraction series by the amplitude dependent tune $Q(J)$, so that the generating function becomes

$$\sigma_\alpha = \frac{-h_\alpha}{i(mQ(J) + q)} + \frac{\sum_{\alpha',\alpha''} \frac{n'm''}{2} \frac{h_{\alpha'}h_{\alpha''}}{(m''Q(J)+q'')}}{i(mQ(J) + q)} + \dots \quad (17)$$

For $mQ(J) + q$ close to zero, we set $\sigma_\alpha = 0$ and obtain the coefficients of the new Hamiltonian function

$$k_\alpha = h_\alpha - \sum_{\alpha',\alpha''} \frac{n'm''}{2} \frac{h_{\alpha'}h_{\alpha''}}{(m''Q(J) + q'')} + \dots \quad (18)$$

[1] There is however a difference between the correction of the tune in the resonance denominator and the correction of the tune obtained from the 2nd order Hamiltonian K. The two terms carry different factors $n'(n'-2)/2$ and $n'(n'+n''-2)/2$ respectively. The physical meaning of this discrepancy is not yet clear

Thus the Hamiltonian function eqn(11) is also modified by replacing Q_0 in the resonance denominator by the action-dependent tune $Q(J)$.

As usual, the sums over the Fourier components q, q', q'' in the expressions for S and K will be carried out analytically leading to

$$S = J\Phi + \sum_{nm} \sigma_{nm} J^{n/2} \frac{sin(m\Phi + \phi_{nm}(J))}{sin(m\pi Q(J))} \quad (19)$$

$$K = \sum_n k_{n00} J^{n/2} + \sum_n k_{nmq} J^{n/2} cos(m\psi + (mQ_0 + q)\theta + \phi_{nmq}) \quad (20)$$

where k_{nmq} is formed by terms like

$$\sum_{n',m',n'',m''} \frac{n'm''}{2\pi} \int_0^L \int_{s'}^{s'+L} ds' ds'' \tilde{h}_{n'm'}(s') \tilde{h}_{n''m''}(s'') \quad (21)$$

$$\times \frac{cos(m''(\phi(s'') - \phi(s') - \pi Q(J)) + m\phi(s') - (mQ_0 + q)\theta')}{sin(m''\pi Q(J))}$$

and $\tilde{h}$ is h in eqn(3) without the exponential. If only one resonant term with $mQ_0 + q \simeq 0$ has to be considered, the transformation

$$\varphi = \psi + (Q_0 + q/m)\theta \quad (22)$$

generated by $F = I\varphi(\psi, s)$

$$A = J;\ \varphi = \psi - \Delta\theta;\ \Delta = Q_0 + q/m$$

leads to the invariant Hamiltonian

$$K = \Delta \cdot A + \sum_n k_{n00} A^{n/2} + \sum_n k_{nmq} A^{n/2} cos(m\varphi + \phi_n). \quad (23)$$

3 Investigation of $4Q_x$-, $6Q_x$- Resonances Driven by Sextupoles

The equations of motion defined by the Hamiltonian have been solved in the case of a linear lattice, which is distorted by sextupole fields. The linear tunes Q_0 are near $Q_x = 0.25$ and $Q_x = 0.1667$. In order to reduce the effort to evaluate eqns(19,20,21), a single sextupole is considered.

The phase space trajectory near the fixed points is obtained by solving eqn(23) for $K = K_0$ with respect to $A = f(K_0, \varphi_0)$ with

$$K_0 = K(A_0, \varphi_0); \quad (24)$$

$$(\partial K/\partial A)_{A=A_0,\varphi=\varphi_0} = (\partial K/\partial\varphi)_{A=A_0,\varphi=\varphi_0} = 0.$$

Because the A-dependent tune is not known a priori, one has to start by using the linear tune in eqn(21) and obtaining a first order correction to the tune by

$$4\pi^2 \Delta Q(A_0) = \oint\oint d\varphi d\theta (\partial K/\partial A)_{A=A_0} \quad (25)$$

Inserting the tune $Q_0 + \Delta Q$ in eqn(21), new values for A_0, φ_0 and K_0 can be evaluated. This procedure converges rapidly and leads to final values $A_0, K_0, \Delta Q(A_0), S(J = A_0)$. Eqn (19) is then used to evaluate the distorted Courant-Snyder invariants and phases I, Φ.

The results for the motion in the vicinity of the 1/4-integer and 1/6-integer resonances are presented in Figs 1,2 which shows phase space plots $\tilde{x}' = x'\beta + x\alpha$ versus x from tracking (dots) and from the analytical treatment (solid lines). Fig 1a shows the result using the Poincaré-Zeipel procedure just above $Q_0 = 0.25$.

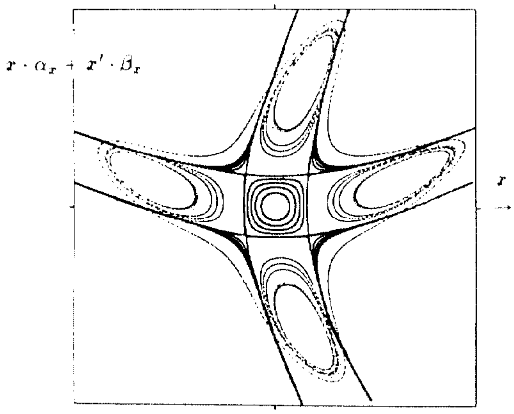

Figure 1: Phase Space Trajectories Near the 1/4 Integer Resonances, Comparison between Tracking (small dots) and the Poincaré-Zeipel Method(solid lines)

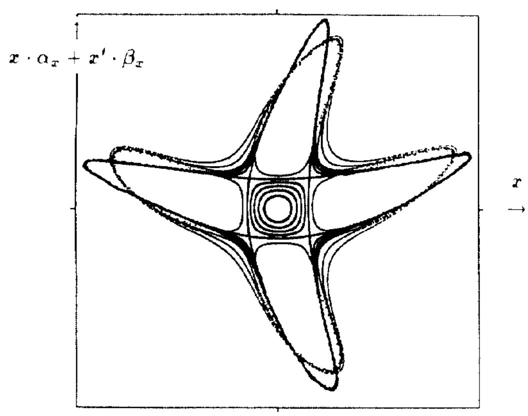

Figure 2: Phase Space Trajectories Near the 1/4 Integer Resonances, Comparison between Tracking (small dots) and the Proposed Method(solid lines)

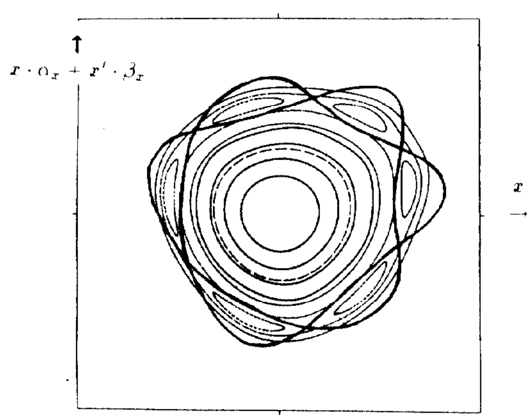

Figure 3: Trajectories Near the 1/6 Integer Resonances, Comparison between Tracking (small dots) and the Poincaré-Zeipel Method(solid lines)

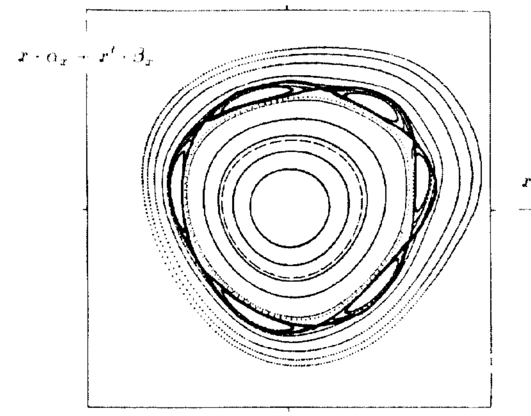

Figure 4: Phase Space Trajectories Near the 1/6 Integer Resonances, Comparison between Tracking (small dots) and the proposed method(solid lines)

Beyond the hyperbolic fixed points, all trajectories are unstable whereas tracking predicts stabilized islands. The refined procedure yields the results shown in Fig1b which shows remarkable agreement between theory and simulation even at extremely large amplitudes. Fig 2 shows the case near the 1/6-integer resonance. This resonance is stable in our lattice because the resonance-driving term k_{46_q} vanishes for $Q_0 = 1/6$. Fig 2a. shows the result of the Poincaré-Zeipel procedure and Fig 2b shows the result of the proposed method for the 1/6 integer resonance island domain. Comparison with simulation shows excellent agreement in contrast to the conventional method.

4 Conclusion

Perturbation theory can be used to predict to high accuracy the phase space structure in the vicinity of high order nonlinear resonances, excited by sextupole fields.

The numerical effort to obtain such results can be rather large because the second order coefficients are the result of a double integral over the lattice. Especially for very high order multipole errors, where many combinations of first order terms combine to higher order effects, the numerical analysis might be as (computer-) time consuming as numerical simulations.

The method should be used for a more realistic estimate of the width of nonlinear resonance islands which in turn might be used to characterize the strength and the potential hazard of nonlinearities. Together with a more accurate calculation of the detuning coefficients, this should improve the analytical prediction of global chaos according to Chirikov's overlap criterion.

References

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