EMITTANCE GROWTH INDUCED BY RADIATION FLUCTUATIONS IN THE FINAL FOCUS OF A LINEAR COLLIDER

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Abstract We investigate the consequences of the quantum fluctuations associated with synchrotron radiation in the dispersive section of the final focus for a 1 TeV linear collider.

A previous estimate [1] of the emittance growth in the bending plane is generalized and used to improve the design of a flat-beam final-focus system. We further discuss the uncompensated chromatic behaviour of the downstream channel, with respect to the quantum induced energy spread.

1 Introduction

The final focus of any future e^+e^- linear collider in the TeV energy range must reduce the transverse dimensions of the two opposing beams by a large factor. Since, after acceleration, the relative energy spread in the beams is at least of a few per mil [2], the luminosity of the collider could seriously be limited by chromatic effects. One way out of this problem is to replace the last quadrupoles by a plasma lens [3], having a very short focal length. A more practical alternative consists in the conventional chromatic compensation scheme, which makes use of magnetic sextupole families located in a dispersive region of the optical channel to 'pre-correct' the chromaticity associated with the last quadrupoles. In order to create the required dispersion, however, the final focus must contain some bending magnets which, owing to the high particle energy, can give rise to significant synchrotron radiation.

In the extreme relativistic regime, a particle with Lorentz factor γ going through a dipole of length L and bending radius ρ undergoes an average relative energy loss $\overline{\Delta E}/E$ given classically by

$$\frac{\overline{\Delta E}}{E} = \frac{2}{3}\gamma^3\frac{r_e L}{\rho^2}, \tag{1}$$

where $r_e = e^2/mc^2$ is the classical electron radius. On the other hand, the average number of photons N_{ph} emitted by the particle is

$$N_{ph} = \frac{5}{2\sqrt{3}}\alpha\gamma\frac{L}{\rho}, \tag{2}$$

with $\alpha = e^2/\hbar c$ the fine structure constant. For a particle energy of 1 TeV, a total dipole length of 80 m and a typical dipole field of 0.1 Tesla [4], corresponding to a bending radius of 3.3×10^4 m, we obtain $\overline{\Delta E}/E \simeq 10^{-3}$ and $N_{ph} \simeq 50$. Since the average number of emitted photons is not very high, the effect of quantum fluctuations can not be neglected: the next section contains a detailed calculation of the resulting increase in the phase-space volume occupied by the particles. In particular, we consider the emittance growth in the bending plane of a flat-beam final focus and the quantum-induced energy spread $(\sigma_E)_q$. The latter can be approximated by

$$(\sigma_E)_q \simeq \frac{2}{\sqrt{N_{ph}}}\overline{\Delta E}, \tag{3}$$

where the factor two is due to the fact that synchrotron radiation mainly consists of 'soft photons', while only a few 'hard photons' are effective in increasing the energy spread [5]. Inserting the previous numerical values into Eq. (3), we obtain a quantum-induced, relative energy spread $(\sigma_E)_q/E \simeq 3 \times 10^{-4}$, which gives rise to uncompensated chromatic aberrations in the downstream optical channel.

It is worth recalling that in a circular machine the effect of quantum fluctuations is counteracted by radiation damping and therefore the beam distribution reaches a steady state after a few relaxation periods. To the contrary, in the final focus of a linear collider there is no damping of the transverse particle oscillations (because RF-cavities are absent and the photon emission is essentially in the same direction as the particle velocity) and longitudinal damping is usually very weak. This can be seen by comparing the reduction of the beam energy spread σ_E caused by radiation damping, which is of the order of $\sigma_E \times \overline{\Delta E}/E$, to the quantum induced energy spread $(\sigma_E)_q$. From Eq. (3) it follows:

$$\frac{\text{radiation damping}}{\text{quantum fluctuations}} \sim \frac{\sigma_E \times \dfrac{\overline{\Delta E}}{E}}{(\sigma_E)_q} \simeq \frac{\sqrt{N_{ph}}}{2}\frac{\sigma_E}{E}. \tag{4}$$

Since the relative energy spread σ_E/E is of the order of a few per mil, while the square root of N_{ph} can hardly approach ten, radiation damping effects can be neglected.

2 Calculation of radiation effects

We consider a magnetic channel whose central design trajectory lies in the horizontal plane. This plane will be referred to as the bending plane and the curvilinear abscissa s along the reference trajectory will be used as the independent variable. Then, in linear approximation, the channel is characterized by the transfer matrix $M(s|s_0)$ which relates the particle phase-space coordinates x_α at s to their initial values at s_0. For simplicity, we assume that there is no coupling between the horizontal and vertical planes: therefore we can limit our analysis to the four-dimensional phase space

$$x_\alpha = (x, x', z, \delta), \qquad \alpha = 1, \ldots, 4 \tag{5}$$

where $x' \equiv dx/ds$ is the slope of the particle trajectory in the bending plane, z is the longitudinal distance from the center of the bunch and $\delta = \Delta E/E$ the relative energy deviation.

The phase-space region occupied by the particle beam around the central trajectory $x_\alpha = 0$ can be described by the 4×4 envelope matrix $R_{\alpha\beta}$ defined by

$$R_{\alpha\beta} = < x_\alpha x_\beta >, \tag{6}$$

where the symbol $< \ldots >$ denotes an average over the beam distribution. Then, starting from the linearized Fokker-Planck equation and neglecting radiation damping effects, it can be shown [6,7] that the evolution of the beam envelope matrix is given by the following expression:

$$R(s) = M(s|s_0)R(s_0)\widetilde{M}(s|s_0) + \int_{s_0}^{s} ds' M(s|s')B(s')\widetilde{M}(s|s'), \tag{7}$$

$$\begin{pmatrix} x_\beta \\ x'_\beta \\ z_s \\ \delta \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}(\cos\phi + \alpha_0 \sin\phi) & \sqrt{\beta_0\beta}\sin\phi & 0 & 0 \\ \frac{(\alpha_0 - \alpha)\cos\phi - (1+\alpha_0\alpha)\sin\phi}{\sqrt{\beta_0\beta}} & \sqrt{\frac{\beta_0}{\beta}}(\cos\phi - \alpha\sin\phi) & 0 & 0 \\ 0 & 0 & 1 & \Gamma \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_\beta \\ x'_\beta \\ z_s \\ \delta \end{pmatrix}_0$$

$$M(s|s_0) = \begin{pmatrix} \sqrt{\gamma_0\beta}\cos(\phi - \psi_0) & \sqrt{\beta_0\beta}\sin\phi & 0 & D - \sqrt{\beta\mathcal{H}_0}\cos(\phi - \theta_0) \\ -\sqrt{\gamma_0\gamma}\sin(\phi - \psi_0 + \psi) & \sqrt{\beta_0\gamma}\cos(\phi + \psi) & 0 & D' + \sqrt{\gamma\mathcal{H}_0}\sin(\phi - \theta_0 + \psi) \\ D'_0 - \sqrt{\gamma_0\mathcal{H}}\sin(\phi - \psi_0 + \theta) & \sqrt{\beta_0\mathcal{H}}\cos(\phi + \theta) - D_0 & 1 & \Gamma + \sqrt{\mathcal{H}\mathcal{H}_0}\sin(\phi - \theta_0 + \theta) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Table 1: Evolution of the normal-mode variables and tranfer matrix for the magnetic channel.

where $\widetilde{M}$ is the transpose of M and the matrix B is given by

$$B_{\alpha\beta} = b(s)\delta_{\alpha 4}\delta_{\beta 4}. \tag{8}$$

The function $b(s)$ describes the effect of the quantum fluctuations associated with synchrotron radiation in the dipole magnets and depends on the bending radius $\rho(s)$ of the reference trajectory

$$b(s) = C_2 \frac{E^5(s)}{|\rho^3(s)|}, \tag{9}$$

$$C_2 = \frac{55}{24\sqrt{3}} \frac{r_e \hbar c}{(mc^2)^6} \simeq 4.13 \times 10^{-11} \frac{\mathrm{m}^2}{(\mathrm{GeV})^5}. \tag{10}$$

It is important to stress that, according to Eq. (7), the evolution of *the beam envelope matrix R obeys a linear superposition principle:* the final envelope matrix R_f at s is the sum of the initial matrix $\widehat{R}_i$ at s_0, propagated by the transfer matrix M, and of a quantum-induced matrix R_q

$$R_f = \widehat{R}_i + R_q. \tag{11}$$

Inserting Eq. (8) into Eq. (7), this quantum-induced contribution to the beam envelope matrix can be written

$$(R_q)_{\alpha\beta} = \int_{s_0}^{s} ds'\, b(s') M_{\alpha 4}(s|s') M_{\beta 4}(s|s'). \tag{12}$$

This formula can easily be generalized to the case of a six-dimensional phase space with horizontal-vertical coupling and has the merit of expressing the effect of radiation fluctuations in terms of the transfer matrix M of the magnetic channel.

For calculation purposes, it is useful to express the matrix M through the optical functions of the channel, i.e. the dispersion $D(s)$ and the beta function $\beta(s)$ in the bending plane. This can be done in two steps: we first introduce the betatron variables x_β, x'_β and the synchrotron displacement z_s, related to the original phase-space variables by the linear transformation

$$\begin{pmatrix} x \\ x' \\ z \\ \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & D \\ 0 & 1 & 0 & D' \\ -D' & D & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_\beta \\ x'_\beta \\ z_s \\ \delta \end{pmatrix}, \tag{13}$$

and then we consider the evolution of these normal-mode variables along the channel, given in Table 1. Here $\alpha = -\beta'/2$, ϕ is the betatron phase advance from s_0 to s and Γ is the corresponding dilation factor of the particle trajectory

$$\phi(s, s_0) = \int_{s_0}^{s} \frac{ds'}{\beta(s')}, \quad \Gamma(s, s_0) = \int_{s_0}^{s} ds' \left[\frac{1}{\gamma^2(s')} - \frac{D(s')}{\rho(s')}\right]. \tag{14}$$

Let us remark that the dispersion funtion $D(s)$ is uniquely defined by requiring that both D^* and D'^* vanish at the interaction point of the collider $s = s^*$. On the contrary, even though $\alpha^* = 0$, the beta function $\beta(s)$ depends on the initial value β_0 or, equivalently, on the final value β^* at the interaction point and therefore it must be considered as associated with the particle beam rather than with the optical channel. Nevertheless, the transfer matrix $M(s|s_0)$, which depends on the optical channel alone, can be expressed in terms of $D(s)$ and $\beta(s)$. After some algebra, it can be cast in the form shown in Table 1, where

$$\begin{aligned} \gamma &= \frac{1+\alpha^2}{\beta}, & \psi &= \arctan(\alpha), \\ \mathcal{H} &= \frac{D^2 + (\alpha D + \beta D')^2}{\beta}, & \theta &= \arctan\left(\frac{\alpha D + \beta D'}{D}\right). \end{aligned} \tag{15}$$

At the interaction point the normal-mode variables coincide with the original variables and, since we also have $\mathcal{H}^* = 0$, $\psi^* = 0$ and $\gamma^* = 1/\beta^*$, from Eq. (12) it follows that the quantum-induced growth of the main beam parameters can be written

$$\begin{aligned} < x_q^2 > &= C_2\beta^* \int_{s_0}^{s^*} ds\, \frac{E^5(s)}{|\rho^3(s)|}\mathcal{H}(s)\cos^2\Phi(s), & (16) \\ < x_q'^2 > &= \frac{C_2}{\beta^*} \int_{s_0}^{s^*} ds\, \frac{E^5(s)}{|\rho^3(s)|}\mathcal{H}(s)\sin^2\Phi(s), & (17) \\ < x_q x_q' > &= -C_2 \int_{s_0}^{s^*} ds\, \frac{E^5(s)}{|\rho^3(s)|}\mathcal{H}(s)\sin\Phi(s)\cos\Phi(s), & (18) \\ < z_q^2 > &= C_2 \int_{s_0}^{s^*} ds\, \frac{E^5(s)}{|\rho^3(s)|}\Gamma^2(s^*, s), & (19) \\ < \delta_q^2 > &= C_2 \int_{s_0}^{s^*} ds\, \frac{E^5(s)}{|\rho^3(s)|}, & (20) \end{aligned}$$

with the phase-angle Φ defined by

$$\Phi(s) = \phi(s*, s) - \theta(s) = \int_{s}^{s^*} \frac{ds'}{\beta(s')} - \arctan\left(\frac{\alpha D + \beta D'}{D}\right). \tag{21}$$

3 Quantum-induced energy spread and emittance growth

For a constant bending radius ρ, the quantum-induced energy spread given by Eq. (20) can be approximated by Eq. (3). The downstream optical channel behaves practically as an uncompensated chromatic system for this additional contribution to the original energy spread of the beam. Indeed, if the synchrotron energy loss of a particle differs from the average as a consequence of statistical fluctuations in the number and energy of the emitted

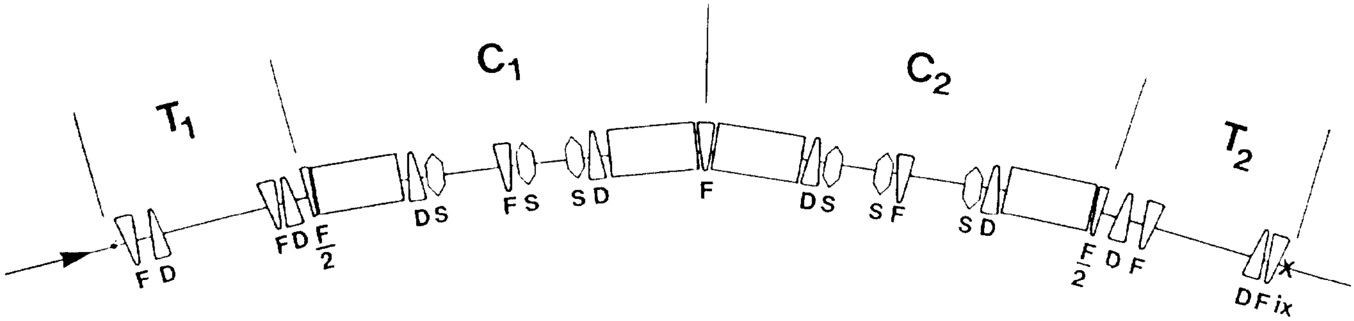

Figure 1: Sketch of the new flat-beam final focus design presented in Ref. 8. Each of the telescopes T_1 and T_2 has a demagnification factor of 6 in the horizontal plane and 18 in the vertical plane, while C_1 and C_2 are symmetric chromatic-correction cells.

photons, the sextupoles do no longer compensate exactly for the energy-dependent focal strength of the subsequent quadrupoles and the effect is more pronounced for those fluctuations which occur after the last sextupole.

A qualitative estimate of the maximum acceptable energy spread induced by quantum fluctuations in a flat-beam final focus can be obtained by requiring that

$$\frac{(\sigma_E)_q}{E} \lesssim \frac{\beta_y^*}{f}, \tag{22}$$

where β_y^* is the vertical beta function at the interaction point and f the focal length of the last quadrupole: this is approximately the chromatic limit for an uncompensated system. In the flat-beam final focus of Ref. 4 we had $\beta_y^* \simeq 1.6$ mm and $f \simeq 5$ m, therefore criterion (22) shows that the quantum-induced, relative energy spread of 3×10^{-4} computed in the introduction is very close to the chromatic limit for the vertical plane. The new flat-beam final focus design presented in Ref. 8 and sketched here in Fig 1 is in a slightly worse situation: it contains four dipole magnets 40 m long with a bending radius of 5×10^4 m and the last quadrupole has a focal length of 3.5 m, yielding a vertical beta function at the interaction point $\beta_y^* \simeq 0.4$ mm. The corresponding chromatic limit is 1.1×10^{-4}, while the quantum-induced energy spread is $(\sigma_E)_q/E \simeq 2.3 \times 10^{-4}$. However we will see that a more serious limitation of this new final focus design is associated with the emittance growth in the bending plane.

Although the quantum-induced growth of the horizontal beam size, given by Eq. (16), is the only relevant parameter affecting the luminosity of the collider, the effect of radiation fluctuations is customarily discussed in terms of emittance growth [1]. To this end, one defines the beam emittance as the square root of the determinant of the betatron envelope matrix. Then, recalling Eq. (11), the final horizontal emittance ε_f is related to its initial value ε_i as follows:

$$\varepsilon_f^2 = \varepsilon_i^2 + \varepsilon_q^2 + \varepsilon_i \varepsilon_p, \tag{23}$$

where ε_q is the quantum-induced emittance for a pencil beam

$$\varepsilon_q^2 = < x_q^2 >< x_q'^2 > - < x_q x_q' >^2, \tag{24}$$

while, using Eqs. (16)-(18), the factor ε_p which appears in the cross-product term can be written

$$\varepsilon_p = \frac{< x_q^2 >}{\beta^*} + \beta^* < x_q'^2 > = C_2 \int_{s_0}^{s^*} ds \, \frac{E^5(s)}{|\rho^3(s)|} \mathcal{H}(s). \tag{25}$$

If the slope of the betatron function at the beginning of a bending magnet is negligible (i.e. $\alpha_0 \simeq 0$) and the betatron phase advance from s_0 to s^* is an integer multiple of π, the quantum contributions to the betatron envelope matrix become

$$< x_q^2 > = C_2 \beta^* \frac{E^5 L^5}{\rho^5 \beta_0} \left[\frac{1}{20} + u \left(u - \frac{1}{3} \right) \right], \tag{26}$$

$$< x_q'^2 > = \frac{C_2}{\beta^*} \frac{E^5 L^3 \beta_0}{\rho^5} \left[\frac{1}{3} + v (v+1) \right], \tag{27}$$

$$< x_q x_q' > = -C_2 \frac{E^5 L^4}{2\rho^5} \left[\frac{1}{4} - u - v \left(u - \frac{1}{3} \right) \right], \tag{28}$$

where

$$u = \frac{D_0 \rho}{L^2}, \qquad v = \frac{D_0' \rho}{L}. \tag{29}$$

In the case of the new final focus design of Ref. 8, D_0 and D_0' refer either to the beginning or to the end of the bending magnets, because of the mirror symmetry of the chromatic-correction cells. Moreover $\beta_0 \simeq 1$ m is much smaller than $L \simeq 40$ m while u and v are either zero (for the outer dipoles) or of order one (for the inner dipoles, where the dispersion is large). Therefore, for an initial horizontal emittance $\varepsilon_i \simeq 10^{-12}$ m, the contribution of ε_q in Eq. (23) is negligible and the final emittance is dominated by the cross-product term, proportional to $\varepsilon_p \simeq < x_q^2 > /\beta^*$: in each of the inner dipoles we have $\varepsilon_p \simeq 8.6 \times 10^{-12}$ m, corresponding to a relative emittance growth of nearly a factor 3. However, owing to the dependence of ε_p on the inverse fifth power of ρ, this growth can be reduced to a more acceptable level for example by halving the bending field in the dipoles and by rematching the sextupole strengths.

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