Transverse Impedance of a Conducting Cylindrical Pipe with Discontinuous Cross Section

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Introduction

A bunch of charged particles travelling in an linear or circular accelerating machine interacts with the surrounding structure inducing electromagnetic fields which, acting back on the bunch, influence its dynamics.

The knowledge of these fields is necessary for the prediction of the energy lost by the beam and for the analysis of the instability phenomena.

In this paper we study the fields induced by a dipole charge distribution crossing a discontinuity. An exact analytical expression of the longitudinal and transverse dipole impedance is derived by extending the analysis developed in Ref.1 for the longitudinal case.

Description of the problem

Let us consider a point charge travelling off-center, with velocity $v=\beta c$, along the z-direction in the structure shown in fig.1. Let $\mathbf{r}_0(r_0,z_0,\theta_0=0)$ be the position of the charge and $\mathbf{r}(r,z,\theta)$ be the observation point in cylindrical coordinates.

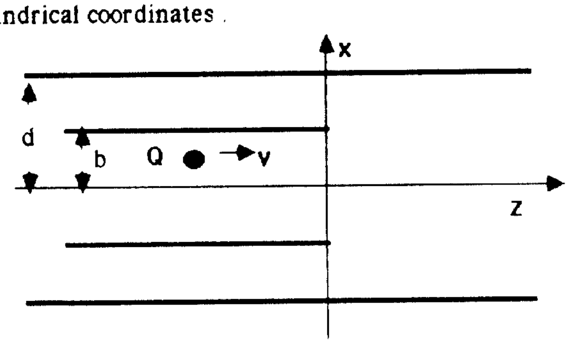

Fig.1 The relevant Geometry

The dipole current density corresponding to a displaced moving unity charge, in the frequency domain ($-i\omega t$), is given by:

$$\mathbf{J}_0^{\delta}(r,z,\theta) = \hat{\mathbf{z}}\,\frac{e^{i\alpha_0 z}}{\pi}\,\frac{\delta(r-r_0)}{r}\cos(\theta) \qquad (1)$$

where $\alpha_0 = \omega/\beta c$, and $\delta(x)$ is the Dirac function.

Fields can be seen as generated by the source current (1) and by the induced currents on the perfectly conducting surrounding walls. The unknown current flowing on the discontinuous pipe can be written as:

$$\mathbf{J}_i^{\delta}(r,z,\theta) = \hat{\mathbf{z}}\,\frac{f(z)}{2\pi^2}\,\frac{\delta(r-b)}{r}\cos(\theta) \qquad (2)$$

where $f(z)$ is the only unknown function of our problem, to be determined.

We first solve the wave equation for the Hertz potential where the forcing terms are the given by the current densities (1) and (2):

$$\nabla^2 \Pi_z^{\delta}(r,z,\theta) + k^2 \Pi_z^{\delta}(r,z,\theta) = \frac{-iZ_0}{k}\left[J_0^{\delta} + J_i^{\delta}\right]$$

where $k=\omega/c$ and Z_0 is the vacuum impedance ($120\pi\,\Omega$). The electric field is related to the Hertz potential by:

$$\mathbf{E}(r,z,\theta) = \nabla(\nabla\cdot\mathbf{\Pi}) + k^2\mathbf{\Pi}$$

Since the currents flow toward the z-direction we have used in eq. (3) the sole z-component of the Hertz potential which is sufficient to derive the fields excited in the pipe complex.. The presence of the outermost pipe is taken into account by imposing the proper boundary conditions $\Pi_z = 0$ at the surface $r=d$:

The linearity of the wave equation allows potential Π_z and fields to be thougth of as sum of two terms corresponding to the source currents (1) and (2).

The electric field due to the dipole charge distribution travelling into a pipe of radius d is [2]:

$$E_{z0}^{\delta}(r,z,\theta) = -\frac{Z_0}{4k}\cos(\theta)\, e^{i\alpha_0 z}\, G^{\delta}(\alpha_0; r, r_0, d)$$

where the function G^{δ}, explicitely given in Appendix, has a twofold expression depending whether the observation point, with coordinates (r,θ) in the transverse circular cross section, is inside or outside the circle of radius r_0.

The electric field created by the induced currents is formally derived in terms of the function $F(\alpha)$ which is the Fourier transform of the unknown function $f(z)$ in the α-domain[2]:

$$E_{zi}^{\delta}(r,z,\theta) = -\frac{Z_0}{8\pi k}\cos(\theta)\int_{-\infty}^{\infty} F(\alpha)\, G^{\delta}(\alpha; r, b, d)\, e^{i\alpha z}\, d\alpha$$

Again the function G^{δ} has a twofold expression in the ranges $r<b$ and $r>b$ respectively.

As the longitudinal component of the electric field must vanish at the surface $r=b$, for $z<0$, and reminding that the induced currents exist only in the region $z<0$, we can write the following integral equations in the α-domain

$$\int_{-\infty}^{\infty} F(\alpha)\, G^{\delta}(\alpha; b,b,d)\, e^{i\alpha z}\, d\alpha = -2\pi i\, G^{\delta}(\alpha_0; b, r_0, d)\, e^{i\alpha_0 z}, \quad z<0$$

$$\int_{-\infty}^{\infty} F(\alpha)\, e^{i\alpha z}\, d\alpha = 0, \qquad z > 0$$

These equations can be solved in the complex α-plain by applying the Wiener-Hopf techniques[3]. A unique solution is obtained by resorting to the "edge condition" which in the α-domain requires:

$$\lim_{|\alpha|\to\infty} F(\alpha) \sim |\alpha|^{-3/2}$$

3. Longitudinal DipoleImpedance

The effects of the longitudinal component of the electric field are examined in the frequency domain by means of the longitudinal coupling impedance. By using the impedance definition, after some manipulations we get:

$$Z^{\delta}_{//}(r,\theta) = -\int_{-\infty}^{\infty}\left[E^{\delta}_{z0}(r,o,\theta) - \frac{I_1(\xi r)}{I_1(\xi b)} E^{\delta}_{z0}(b,o,\theta)\, u(-z)\right] dz + i\,\frac{E^{\delta}_{z0}(b,o,\theta)}{G^{-}(\alpha_0)}\left[\frac{\partial}{\partial\alpha}\frac{G(\alpha;r,b,d)}{G^{+}(\alpha)}\right]_{\alpha=\alpha_0}$$

where $\xi = k/\beta\gamma$ and $G^{\pm}(\alpha)$ are auxiliar functions

In our calculations we generalized the expression of the impedance by keeping as distinct the location of the source from the point where the fields are observed.

In the analysis of the above expression we recognizes two contributions: a term due to a dipole charge distribution moving within an infinite pipe without discontinuities, and an additional impedance due to the step.

The impedance per unit length (Ω/m) of the infinite pipe merges into the usual expression when we assume the source and test points to be coincident; in the case $\xi d \ll 1$ we get the following approximation:

$$\frac{\partial Z^{\delta}_{//}(r_0, o)}{\partial z} \sim \frac{iZ_0 k}{4\pi(\beta\gamma)^2}\begin{cases} 1-\left(\dfrac{r_0}{d}\right)^2 & z>0 \\[2ex] 1-\left(\dfrac{r_0}{b}\right)^2 & z<0 \end{cases}$$

The impedance term accounting for the step in the pipe wall is found to be

$$Z^{\delta}_{//}(r,\theta) = \frac{ik}{\beta\xi^2} E^{\delta}_{z0}(b,o,\theta)\,\frac{I_1(\xi r)}{I_1(\xi b)}\left[X + \xi r\frac{I_0(\xi r)}{I_1(\xi r)}\right]$$

with

$$X = (1+\beta) - \xi b\frac{I_0(\xi b)}{I_1(\xi b)} - \frac{\xi}{\gamma}\Sigma$$

where the sum Σ is explicitely given in Appendix. The above expression vanishes on the axis, and obviously when b=d (no step) and $r_0=0$ (no dipole term in the charge distribution). For $r=r_0$, the real part, related to energy lost, and the imaginary part are shown in Figs.2 versus the normalized frequency kb, for two energy values.

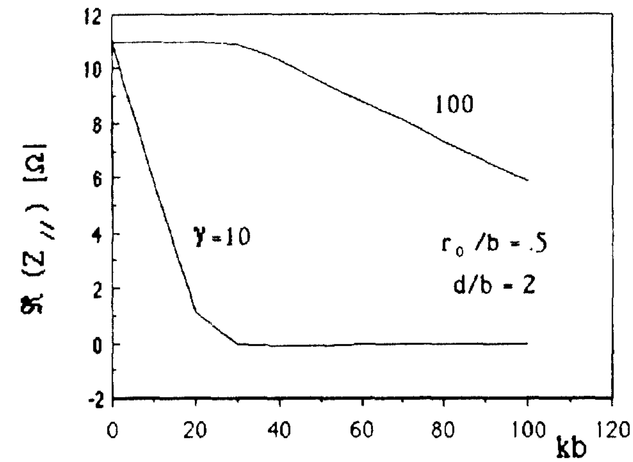

Fig. 2a : Real part of Longitudinal Dipole Impedance

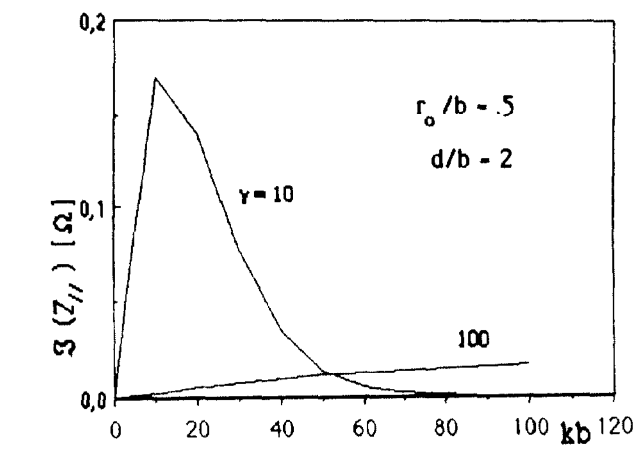

Fig.2b Imaginary part of Longitudinal Dipole Impedance

In the approximation $\xi d \ll 1$ we get the simple useful expression:

$$Z^{\delta}_{//}(r,\theta) \sim \frac{Z_0}{2\pi}\cos(\theta)\,\frac{r_0 r}{b^2}\left[1-\left(\frac{b}{d}\right)^2\right]$$

4. Transverse Dipole Impedance

Applying the Panofsky-Wenzel relations existing between the longitudinal and transverswe force:

$$F^{\delta}_{\perp} = \frac{-i}{k}\nabla_{\perp} F^{\delta}_{//}$$

we may express the transverse impedance in terms of the longitudinal one as follows:

$$Z^{\delta}_{\perp}(r,\theta) = \frac{1}{kr_0}\nabla_{\perp} Z^{\delta}_{//}(r,\theta)$$

As for the longitudinal impedance, we obtain an impedance term due to an infinite pipe and an additional contribution due to the discontinuity.
In the approximation $\xi d \ll 1$, the former term merges into the well known one :

$$Z_\perp^\delta(r_0, o) \approx \frac{iZ_0}{2\pi(\beta\gamma)^2}\left\{\begin{array}{ll} \dfrac{1}{r_0^2}-\dfrac{1}{d^2} & z>o \\ \dfrac{1}{r_0^2}-\dfrac{1}{b^2} & z<o \end{array}\right\}$$

which is directed along the dipole direction $\mathbf{r_0}$ and therefore represented as a scalar.
The general expression of the latter term's two components are:

$$Z_\theta^\delta(r,\theta) = \frac{-Z_0\sin(\theta)}{2\pi\beta k}\,\frac{I_1(\xi r_0)}{r_0}\,\frac{I_1(\xi r)}{r}\left[\frac{K_1(\xi b)}{I_1(\xi b)}-\frac{K_1(\xi d)}{I_1(\xi d)}\right]\left[X+\xi r\frac{I_0(\xi r)}{I_1(\xi r)}\right]$$

$$Z_r^\delta(r,\theta) = \frac{Z_0\cos(\theta)}{2\pi\beta^2\gamma}\,\frac{I_1(\xi r_0)}{r_0}\left[\frac{K_1(\xi b)}{I_1(\xi b)}-\frac{K_1(\xi d)}{I_1(\xi d)}\right]\left\{I_0(\xi r)(1+X)+\frac{I_1(\xi r)}{\xi r}\left[(\xi r)^2-X\right]\right\}$$

For $\theta=0$ the transverse impedance has only the radial component.; its behaviour is shown in Figs.3 in the case $r=r_0$

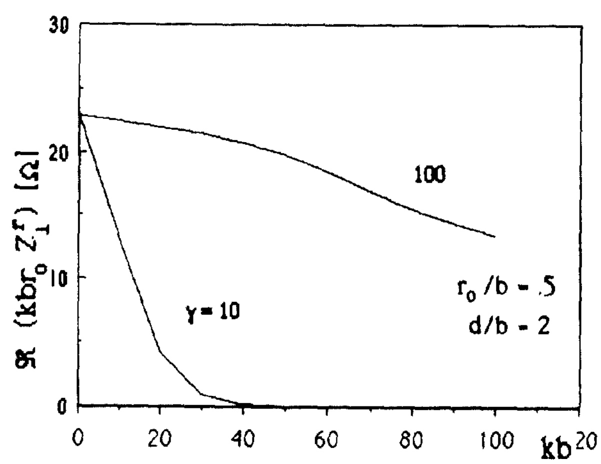

Fig.3a Real part of Transverse Impedance: r component

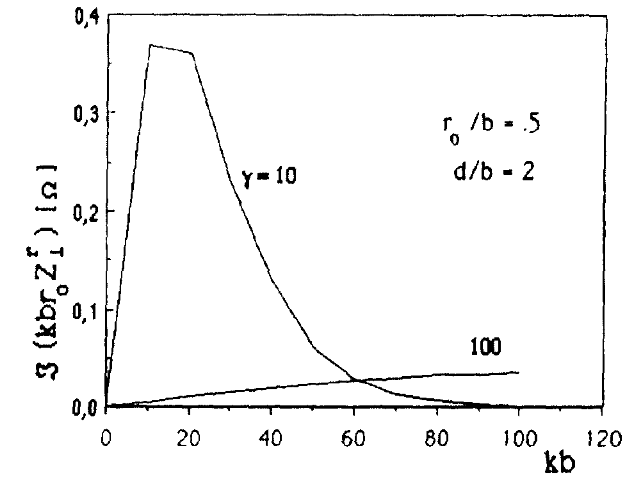

Fig. 3b Imag. part of Transverse Impedance: r component

In the approximation $\xi d \ll 1$ the transverse impedance turns out to be directed along $\mathbf{r_0}$ and has the simple expression:

$$Z_\perp^\delta = \frac{Z_0}{2\pi k b^2}\left[1-\left(\frac{b}{d}\right)^2\right]$$

Appendix

$$G^\delta(\alpha;r,r',d) = \frac{\Omega^2}{J_1(\Omega d)}\left\{\left[\begin{array}{c} J_1(\Omega r)\,H_1(\Omega r')\,J_1(\Omega d) \\ J_1(\Omega r')\,H_1(\Omega r)\,J_1(\Omega d)\end{array}\right] - J_1(\Omega r)\,H_1(\Omega d)\,J_1(\Omega r')\right\}$$

where $\Omega = (k^2-\alpha^2)^{1/2}$, $J_1(x)$ and $H_1(x)$, are Bessel functions.

$$\Sigma = \sum_{n=0}^{\infty}\left(\frac{1}{\alpha_n^b-\alpha_0}-\frac{1}{\alpha_n^d-\alpha_0}+\frac{1}{\alpha_n^c-\alpha_0}\right)$$

where α_n^b, α_n^d and α_n^c are the zeroes of $J_1(\Omega b)$, $J_1(\Omega d)$ and of the cross product $J_1(\Omega b)H_1(\Omega d)-J_1(\Omega d)H_1(\Omega b)$ respectively.

References

[1] L.Palumbo, "Analytical Calculation of the Impedance of a Discontinuity", to be published on Particle Accelerators.
[2] E.Gianfelice, L.Palumbo, "Transverse Impedance of a Discontinuity", in prep.
[3] L.A.Weinstein,"The Theory of Diffraction and the Factorization Method", The Golden Press, Boulder, Colorado, 1969.