J.M. Jowett, A. Hofmann

CERN, 1211 Geneva 23, Switzerland

K.L.F. Bane, P.L. Morton, W. Spence, R. Stege

Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305

Abstract Two types of beam response measurements were carried out in SPEAR with the aim of gaining understanding of high intensity phenomena. In the first case, the beam was kicked horizontally with the injection kicker magnet and the amplitude of the resulting betatron oscillation was observed as a function of time. This experiment was carried out for different beam intensities. At large currents, the decay of the oscillation is dominated by the "head-tail damping" due to the transverse impedance. It shows the expected dependence on the chromaticity. At small currents, the decay of the excited betatron oscillation is due mostly to Landau damping. Another series of experiments dealt with the longitudinal behaviour of a single bunch. A network analyser was used to modulate the phase or amplitude of the RF voltage and thereby excite dipole or quadrupole synchrotron oscillations. The modulating frequency was swept through the dipole or quadrupole frequency and the beam response in phase and amplitude, the longitudinal bunched-beam transfer function, was observed. The behaviour of the transfer function changed significantly as the beam current was increased. We outline a theoretical framework for the interpretation of these measurements.

1 Head-tail damping

Using a broad-band model of the impedance of SPEAR (resonant frequency $\omega_r/2\pi \simeq 1.3$ GHz, Q = 1) one can derive the expected head-tail damping rate

$$\frac{1}{\tau} = \frac{c^2 e \bar{I} |Z_{10}| \xi_x Q_x}{4\sqrt{2} E \sigma_z \alpha \omega_\tau},\tag{1}$$

where $\alpha \simeq 0.042$ is the momentum compaction, Q_x is the horizontal betatron tune, $\sigma_z \simeq 50 \text{ mm}$ is the bunch length, \bar{I} is the average bunch current and $\xi_x = pdQ_x/dp$ is the chromaticity.

To measure this damping rate, the beam was kicked (horizontally) with the injection kicker and an exponential decay of the resulting coherent betatron oscillation was observed on a fast oscilloscope. We made measurements of the horizontal damping rate both as a function of current \bar{I} at fixed ξ_x and with fixed \bar{I} and varying ξ_x (see Figure 1). At low currents the decay of the oscillation is dominated by Landau and radiation damping.

Fitting the data in Figure 1(a) to the formula (1) gives $|Z_{\perp 0}| \simeq 0.18 \,\mathrm{M}\Omega$ while the data of Figure 1(b) yield $|Z_{\perp 0}| \simeq 0.24 \,\mathrm{M}\Omega$. These results agree quite well with each other and also with similar measurements taken later when the "SPEAR capacitor" [1] had been installed—from which we may infer that this device did not change Z_{\perp} .

Estimating the longitudinal impedance |Z/n| via the relation $Z_{\perp} \simeq (2R/b^2)|Z/n|$ where $b \simeq 45 \,\mathrm{mm}$ is an effective chamber radius and $R \simeq 37.4 \,\mathrm{mm}$ is the machine radius, we get $|Z/n| \simeq 5-6 \,\Omega$. This should be considered as an effective value valid for the range of bunch lengths during our measurements.



Figure 1: Head-tail damping rate, (a) at constant chromaticity, (b) at constant current, with E = 2.07 GeV, $\hat{V} = 0.83 \text{ MV}$.

The decay of vertical coherent oscillations was also studied and led to similar results.

2 Longitudinal bunched-beam transfer function

In a further series of experiments we excited dipole and quadrupole synchrotron oscillations of the bunch by modulating the phase or amplitude of the RF voltage. A network analyser was used to sweep the modulating frequency ω through a range encompassing one of the corresponding mode frequencies and the beam response was fed back into its input. The instrument could then display the beam's response to excitation in both phase and amplitude which we call the *longitudinal bunched-beam transfer function* (BBTF). Some representative results are shown in Figure 4.

Transfer functions have been treated for coasting beams [2] and there have been some discussions of the bunched-beam case [3,4,5] although a complete theory in the presence of impedances producing frequency shifts and bunch-lengthening effects is still lacking. Here we present a theoretical model which reproduces, at least qualitatively, the main features of our measurements within a range of beam currents below the threshold of turbulent bunch-lengthening.

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Figure 2: Computed synchrotron frequency distributions.

Figure 3: Computed BBTF: amplitude and phase of dipole mode.

Figure 4: Measured BBTF: amplitude (log scale) and phase of dipole mode.

Theoretical model

Equations of motion Let us describe synchrotron motion using the fractional energy deviation $\varepsilon = \Delta E/E$ and the RF phase deviation ϕ . For simplicity, we make the approximation that the stable phase angle $\phi_s = 0$ and consider the effects of a purely inductive coupling impedance $|Z/n| = \omega_0 L$. Let $f_0 = \omega_0/2\pi$ denote the revolution frequency, \hat{V} , the peak RF voltage and $\omega_{s0} = \omega_0 \sqrt{h\alpha \hat{V} \cos \phi_s/2\pi E} = \omega_0 Q_{s0}$, the angular synchrotron frequency in the linear part of the RF waveform, as distinct from the general amplitude-dependent frequency ω_s . Similarly we distinguish the instantaneous beam current $I_0(\phi)$ for a linear RF waveform from the real instantaneous current $I(\phi)$.

The equations of motion are

$$\dot{\varepsilon} = \frac{\omega_0}{2\pi E} \left[e\hat{V}\sin\phi + eh \left| \frac{Z}{n} \right| \frac{dI}{d\phi} \right], \qquad \dot{\phi} = -\omega_0 h\alpha\varepsilon, \quad (2)$$

where $h = \omega_{\rm rf}/\omega_0$. In the self-consistent field approximation the single-particle Hamiltonian

$$\mathcal{H} = \frac{\omega_0 h \alpha \varepsilon^2}{2} + \frac{Q_{s0}}{h \alpha}^2 \left\{ (1 - \cos \phi) + \frac{h}{\hat{V}} \left| \frac{Z}{n} \right| [I(\phi) - I(0)] \right\}, \quad (3)$$

is a constant of the motion, taking values H.

Phase space distributions In a stationary state, the phase space distribution function $\psi(\varepsilon, \phi)$ should be a function of H only; moreover we know that it is gaussian in ε . Therefore

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$$\psi(\varepsilon,\phi) = C \exp\left\{-\frac{\varepsilon^2}{2\sigma_{\varepsilon}^2} - \frac{1-\cos\phi + \frac{h}{\hat{V}} \left|\frac{Z}{n}\right| [I(\phi) - I(0)]}{\sigma_{\phi}^2}\right\},\tag{4}$$

with $\sigma_{\phi} = h \alpha \sigma_{\varepsilon} / Q_{s0}$ and an appropriate normalisation constant C. Using the relation $I(\phi) = 2\pi h \bar{I} \int \psi(\varepsilon, \phi) d\varepsilon$, we get an implicit equation for the current distribution:

$$I(\phi) \exp\left(\frac{\xi I(\phi)}{I_0(\phi)}\right) = I(0) \exp\left(\frac{\xi I(0)}{I_0(0)}\right) \exp\left(-\frac{1-\cos\phi}{\sigma_{\phi}^2}\right).$$
 (5)

We have defined the dimensionless parameter

$$\xi \stackrel{\text{def}}{=} \frac{\sqrt{2\pi} h^2 \bar{I} |Z/n|}{\hat{V} \sigma_{\phi}^3} \,. \tag{6}$$

Approximating $\phi \ll 1, \xi \lesssim 1$, we obtain the current,

$$I(\phi) = \frac{\sqrt{2\pi h \bar{I}}}{\sigma_{\phi}} e^{-\phi^{2}/2\sigma_{\phi}^{2}} \times \left[1 - \frac{\sigma_{\phi}^{2}}{3} \left(1 - \frac{\phi^{4}}{3\sigma_{\phi}^{4}}\right) - \frac{\xi}{2} \left(2e^{-\phi^{2}/2\sigma_{\phi}^{2}} - \sqrt{2}\right)\right],$$
(7)

which lets us write (4) to first order in ξ as

$$\psi(\varepsilon,\phi) = \frac{1}{2\pi\sigma_{\varepsilon}\sigma_{\phi}} e^{-\varepsilon^{2}/2\sigma_{\varepsilon}^{2}-\phi^{2}/2\sigma_{\phi}^{2}} \times \left[1 - \frac{\sigma_{\phi}^{2}}{3}\left(1 - \frac{\phi^{4}}{3\sigma_{\phi}^{4}}\right) - \frac{\xi}{2}\left(2e^{-\phi^{2}/2\sigma_{\phi}^{2}} - \sqrt{2}\right)\right].$$
(8)

Nonlinear detuning To first order in ξ and neglecting all but the fundamental harmonic, the amplitude-dependence of the synchrotron frequency is found to be (see also [6])

$$\omega_{\bullet}(\eta) = \omega_{\bullet 0} \left[1 - \frac{\eta \sigma_{\phi}^2}{8} - \frac{\xi}{2} e^{-\eta/2} \left(I_0(\eta/2) - I_1(\eta/2) \right) \right], \quad (9)$$



Figure 5: Synchrotron tune dependence on η .



Figure 6: Computed quadrupole mode BBTF (arbitrary units)



Figure 7: Measured quadrupole mode BBTF (arbitrary units)

where $\eta = H/\omega_0 h \alpha \sigma_e^2 = H/\langle H \rangle$ is a "scaled Hamiltonian" variable; I₀ and I₁ are modified Bessel functions.

This function is plotted in Figure 5 for SPEAR parameters and 3 different values of ξ . For $\xi < \sigma_{\phi}^2/3$, the equation $\omega_s(H) = \omega$ has a unique or no solution for H depending on the value of ω . For $\xi > \sigma_{\phi}^2/3$, a bifurcation occurs: $\omega'_s(\eta)$ has a zero and the equation may have 0, 1 or 2 solutions.

Transfer function As a first step we use the equations of motion to find the time-evolution of the beam distribution after a sudden increase in energy: $\varepsilon \mapsto \varepsilon + \delta \varepsilon$ at time t_0 . The resulting dipole moment is evaluated as an appropriate phase space average. The next step is the extension to the case of an oscillating kick applied once per revolution (treated as having a frequency ω in smooth approximation). With the first order approximation to $\psi_0(H)$ we obtain a dipole response with amplitude

$$|D| = \frac{1}{2} \frac{|\tilde{\phi}|}{\omega_{s0}} \int \frac{\eta e^{-\eta} \, d\eta}{\omega_s(\eta) - \omega} \stackrel{\text{def}}{=} \frac{|\tilde{\phi}D_0|}{\omega_{s0}}, \tag{10}$$

where $|\bar{\phi}|$ is the amplitude of phase acceleration due to external and induced fields. We note that the integral in (10) has a real principal part and imaginary contributions from up to 2 poles on the ω -axis. Assuming the oscillation is of the form $e^{-i\omega t}$ with $\omega \simeq \omega_{s0}$, the measured response should be $|\dot{D}| = -i\omega_{s0}|D|$.

To include the force due to fields induced by the coherent motion we calculate the average voltage seen by the bunch,

$$\bar{V}_1 \simeq \int \frac{V_{\text{ind}}(\phi) I_0(\phi) d\phi}{2\pi h \bar{I}} = -\frac{\xi \hat{V} |D|}{2\sqrt{2}},\tag{11}$$

which induces an acceleration $|\tilde{\phi}_{ind}| = (-i\omega_{s0}\xi/2\sqrt{2})\dot{D}$. Finally, the self-consistent (dipole-mode) BBTF is

$$\frac{\dot{D}}{|\ddot{\phi}_{\text{ext}}|} = \frac{1}{\frac{1}{D_0} + \frac{i\omega_{s0}\xi}{2\sqrt{2}}} \,. \tag{12}$$

The results of numerical computations using SPEAR parameters close to those obtaining when we made the measurements are shown in Figure 3. Because of bunch-lengthening, the value of ξ does not increase much with *I*. The upper limit of the integral in (10) is determined by the RF bucket height. Qualitatively the model reproduces the behaviour of the phase and the characteristic double hump in the amplitude of the BBTF which we found when the current was sufficiently large. However it is clear that improvements are desirable, e.g. pushing the theory to higher orders, including a proper representation of the non-gaussian nature of the distribution in the longitudinal coordinate. We intend to pursue this in future work.

Computations of a quadrupole mode at zero current are shown with a corresponding low-current measurement in Figures 6 and 7.

References

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