

ANALYTIC THEORY OF SYNCHRO-BETATRON RESONANCES

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Summary

The coupling mechanisms of synchrotron and betatron motion have been intensively studied in the past by various authors. Piwinski and Wrulich [1] analysed firstly the coupling effect caused by a non-vanishing dispersion in the accelerating cavities. They also found that not only the betatron motion is influenced by the synchrotron motion but also the opposite effect, namely a perturbation of the synchrotron motion via the path lengthening effect takes place. However, their theory is restricted to the linear sideband effects. Suzuki [2] developed a canonical perturbation theory for the nonlinear Hamiltonian describing synchro-betatron resonances induced by dispersion. This theory is based on the formalism of Corsten and Hagedorn [3]. In this contribution we develop a theory of synchro-betatron motion influenced by dispersion in cavities and longitudinal wakefields. This theory starting from the coupled nonlinear differential equations for the vertical betatron motion and the synchrotron motion is not only valid in the neighbourhood of resonances but for any values of the synchrotron and betatron tunes. After introducing the coupled equations we succeed to decouple the synchrotron equation from the betatron equation by neglecting the path lengthening effect. This is a valid approximation since the longitudinal emittance is usually much larger than the transverse one. After this decoupling we solve the synchrotron equation of motion by linearization. Inserting the so obtained result into the betatron equation finally leads to a forced linear oscillator. The excitation coefficients for the resonances due to dispersion in cavities and longitudinal wakefields are then calculated in closed form.

1. Equations of motion

In electron machines usually the vertical emittance of a beam is smaller than the horizontal one. Thus the influence of synchrotron oscillations on the vertical betatron motion is stronger than on the horizontal motion. So in a first step we evaluate the coupling effects in this plane only. The equations read as :

$$y_i'' + (1 - \delta_i) [K(\phi)y_i + K'(\phi) x_i y_i] = \frac{N}{Y} \sum_k \sum_{j=1} \delta_p(\phi - \phi_k) y_j r_e R^2 F(t_i - t_k) \quad (1)$$

$$\delta = \sum_k e V_k \delta_p(\phi - \phi_k) \left[\sin(\psi_s + \Delta\psi) - \frac{h n_y y_i'}{R^2} + \frac{h n' y_i}{R} \right] - \sin\psi_s$$

$$- 2\pi R e^2 \frac{N}{Y} \sum_k \sum_{j \neq i} \delta_p(\phi - \phi_k) G(t_i - t_j) - 2\pi R e^2 \frac{N}{Y} \sum_k \sum_{j \neq i} \delta_p(\phi - \phi_k) y_i y_j H(t_i - t_j) \quad (2)$$

$$\Delta\psi = \alpha h \delta_i \quad (3)$$

The independent variable ϕ is the angle in radians along the machine ring. Dots mean differentiations with reference to ϕ . By y_i we denote the vertical coordinate of the i th path particle in the bunch while δ_i means the relative energy deviation of this

particle at a given angle. By K and K' we describe the quadrupole and sextupole distributions in the ring. R is the machine radius, r_e the classical electron radius, N the number of particles per bunch and γ the usual energy Lorentz factor. Finally F denotes the transverse wake function and t_i the arrival time of the i th particle. The momentum compaction factor is written as α while the dispersion function in our notation is n . Note that ' means differentiation with reference to the lengths s around the machine ring. The voltage of the cavity and the elementary charge are indicated by V and e respectively. G and H describe the monopole and dipole longitudinal wake functions. Finally $\Delta\phi_i$ and ψ_s denote the RF phase of the i th particle and the synchronous phase, respectively. The quantity h represents the harmonic number.

2. Simplifying assumptions

The above coupled equations (1)-(3) are the general ones for the coupled synchrotron-betatron motion. In this contribution, however, we make some simplifying assumptions :

(a) For the transverse motion we neglect sextupoles and wakefields. These terms will be treated in future papers.

(b) For the synchrotron motion we neglect the path-lengthening effect due to the betatron oscillations. This is a valid approximation if the dispersion and its derivative are not too large because the longitudinal emittance is usually much larger than the transverse one. This assumption has also been confirmed by using computer simulations.

(c) We neglect the longitudinal dipole wakefunction H and keep only the monopole wakefunction G , since the former is usually small [4].

(d) We assume that the synchrotron oscillation is linear and thus described by the harmonic oscillator solution :

$$\Delta\psi_i = a \cos(Q_s \phi + \phi_0) \quad (4)$$

After these simplifications and by applying the usual Courant and Snyder transformation [5] for the transverse coordinate $y_{\beta i}$:

$$y_{\beta i} = \beta^{1/2} Y; \quad d\theta = \frac{R d\phi}{Q_y \beta} \quad (5)$$

the final equation for the reduced betatron coordinate Y reads as :

$$\frac{d^2 Y}{d\theta^2} + Q_y^2 Y = - \frac{Q_y^2 \beta^{3/2}}{R^2} \left[n' Y \frac{d\delta}{d\phi} + \frac{d}{d\phi} (n_y \frac{d\delta}{d\phi}) \right] \quad (6)$$

while the equation for the momentum deviations δ_i becomes :

$$\frac{d\delta_i}{d\phi} = \sum_k \frac{eV}{E} \delta(\phi - \phi_k) [\sin(\psi_s + \Delta\psi_i) - \sin\psi_s] - 2\pi R e^2 \sum_k \sum_{j \neq i} \delta(\phi - \phi_k) G(\Delta\psi_i - \Delta\psi_j) \quad (7)$$

By $y_{\beta i}$ we mean the betatron-part of the transverse motion y_i according to the decomposition :

$$y_i = y_{\beta i} + \delta y_i \quad (8)$$

The wakefunction G may be written as [6] :

$$G(t) = \frac{1}{2\pi R} \int_{-\infty}^{+\infty} \exp(-i\omega t) Z_L(\omega) d\omega \quad (9)$$

Knowing the longitudinal impedance $Z_L(\omega)$ together with Eqs. (9), (4) and (7), the right hand side of the equation for the reduced betatron coordinate Y (Eq. (7)) is fully known and thus our problem reduces to solving a linear forced oscillator.

4. Solution of the equation for Y

In order to obtain the final result for $Y(\theta)$ in terms of trigonometric functions with resonance denominators we have to expand the dispersion part as well as the wakefunction part of Eq. (7) into Fourier series with reference to ϕ . Together with Eq.(4) and using elementary properties of Besselfunctions we find :

$$\sin(\Psi_S + \Delta\Psi) =$$

$$\sin\Psi_S [J_0(a) + 2 \sum_{k=1}^{\infty} (-1)^k J_{2k}(a) \cos(2k(Q_S\phi + \phi_0))] +$$

$$+ 2\cos\Psi_S \left[\sum_{k=0}^{\infty} (-1)^k J_{2k+1}(a) \cos((2k+1)(Q_S\phi + \phi_0)) \right] \quad (10)$$

The wakefunction part can be expressed by replacing the sum over the j -th particle by an integral using the normalized charge density $\rho(t, \phi)$. The wakefunction part is denoted by W :

$$W = 2\pi R e \sum_k \delta(\phi - \phi_k) \int_{-\infty}^{\tau} G(t-t') \rho(t', \phi) dt' \quad (11)$$

In case of electron machines we assume a Gaussian bunch :

$$\rho(t, \phi) = \frac{Ne}{2\pi\sigma_t} \exp[-t^2/(2\sigma_t^2)] \quad (12)$$

We further assume a broad-band-resonator impedance :

$$Z_L(\omega) = \frac{R_L}{1 - iQ[\omega/\omega_r - \omega_r/\omega]} \quad (13)$$

Here R_L is the peak value of the longitudinal impedance, Q the quality factor and $\omega_r/(2\pi)$ the resonance frequency. From Eqs. (9) and (11) and the Fourier transform of ρ :

$$\rho(t, \phi) = \int_{-\infty}^{\infty} \tilde{\rho}(\omega, \phi) \exp(-i\omega t) d\omega \quad (14)$$

we obtain the following Fourier representation of W :

$$W = \lambda \sum_k \delta(\phi - \phi_k) \sum_{m=0}^{\infty} A_m \cos[m(Q_S\phi + \phi_0)] \quad (15)$$

where λ is a constant given by :

$$\lambda = \frac{Ne^2 R_L}{2\pi E \sqrt{4Q^2 - 1}} \quad (16)$$

Here we have made use of $t = \Delta\Psi/(\hbar\omega_0)$ where ω_0 is the revolution frequency. For $\Delta\Psi$ we used Eq. (4). The coefficients A_m are given by :

$$A_m = \int_0^{2\pi} d\phi \cos m\phi \left[\exp[-a^2 \cos^2\phi/(\hbar\omega_0\sigma_t)^2] \right. \\ \omega_1 [w(2^{-1/2}(\omega_1\sigma_t + \frac{ia\cos\phi}{\hbar\omega_0\sigma_t})) - 2\exp(-\frac{1}{2}(\omega_1\sigma_t + \frac{ia\cos\phi}{\hbar\omega_0\sigma_t})^2)] \\ - \omega_2 [w(2^{-1/2}(\omega_2\sigma_t + \frac{ia\cos\phi}{\hbar\omega_0\sigma_t})) - 2\exp(-\frac{1}{2}(\omega_2\sigma_t + \frac{ia\cos\phi}{\hbar\omega_0\sigma_t})^2)] \\ - \omega_1 [w(2^{-1/2}(\omega_1\sigma_t) - 2\exp(-\frac{1}{2}(\omega_1\sigma_t)^2))] \\ \left. + \omega_2 [w(2^{-1/2}(\omega_2\sigma_t) - 2\exp(-\frac{1}{2}(\omega_2\sigma_t)^2))] \right] \quad (17)$$

The complex quantities ω_1 and ω_2 are defined as :

$$\omega_{1,2} = +\omega_r \sqrt{4Q^2 - 1}/(2Q) \pm i\omega_r/(2Q) \quad (18)$$

while the complex error-function as usual has been denoted by w . Finally we may expand the periodic δ -function in Eq. (15) :

$$\delta(\phi - \phi_k) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \exp[in(\phi - \phi_k)] \quad (19)$$

Now the complete right hand side of the Y -equation (6) is expressed in terms of trigonometric functions. So the practical solution of Eq. (6) becomes easy and can be evaluated as linear combination of the same functions. We write here the solution for one cavity located at the Courant and Snyder angle $\theta = 0$. The Y_n are defined as $Y(2\pi n + \epsilon)$ (= at the exit of the thin cavity).

$$Y_n = Y_0 \cos 2\pi n Q_y + \frac{Y_0'}{Q_y} \sin 2\pi n Q_y + \\ + \frac{1}{2} \beta_y^{-1/2} (\beta_y \eta_y' + \alpha_y \eta_y) B_1 - \frac{1}{2} \beta_y^{-1/2} \eta_y B_2 \quad (20)$$

$$\left[\frac{1}{Q_y} \frac{dY}{d\theta} \right]_n = -Y_0 \sin 2\pi n Q_y + \frac{Y_0'}{Q_y} \cos 2\pi n Q_y$$

$$- \frac{1}{2} \beta_y^{-1/2} (\beta_y \eta_y' + \alpha_y \eta_y) B_2 - \frac{1}{2} \beta_y^{-1/2} \eta_y B_1 \quad (21)$$

The expressions B_1 and B_2 contain the excitation terms for synchro-betatron resonances due to dispersion and wakefield effects.

$$B_{1,2} = \left[\frac{eV}{E} \sin \Psi_S (J_0(a) - 1) + \lambda/2 A_0 \right] \frac{\cos(2\pi n Q_y - \pi Q_y) - \cos \pi Q_y}{\sin \pi Q_y}$$

$$+ \sum_{m=1}^{\infty} \left[\frac{eV}{E} \alpha_m J_m(a) + \lambda/2 A_m \right]$$

$$\left[\frac{\cos[2\pi n Q_y - \pi(Q_y - m Q_S) + m\phi_0] - \cos[2\pi m Q_S - \pi(Q_y - m Q_S) + m\phi_0]}{\sin \pi(Q_y - m Q_S)} \right] \quad (22)$$

$$+ \frac{\cos[2\pi n Q_y - \pi(Q_y + m Q_S) - m\phi_0] - \cos[2\pi m Q_S + \pi(Q_y - m Q_S) + m\phi_0]}{\sin \pi(Q_y - m Q_S)}$$

The α_y , β_y , η_y and η'_y in Eqs. (20) and (21) are the usual Twiss parameters at the cavity position while α_m in Eq. (22) is :

$$\alpha_m = \begin{cases} (-1)^{m/2} \sin \psi_s & \dots\dots\dots (m : \text{even}) \\ (-1)^{(m-1)/2} \sin \psi_s & \dots\dots\dots (m : \text{odd}) \end{cases} \quad (23)$$

We applied the above analytic results to the case of LEP with one single cavity. The value of Q_s has been chosen to 0.09 while the fractional part of the vertical tune is 0.20. The formulae have been evaluated using the program code SYBILLE written by the authors of this contribution [7]. In Fig. 1 the betatron phase space ($y, dy/d\theta$) at $\theta = (0+\epsilon)$ (cavity exit) is plotted for the first 2000 points. The vertical dispersion has been fixed to a value of $\eta_y = 5$ cm. In this case wakefields have not been taken into account. We observe clearly the existence of an island type motion around the fifth order fixed points caused by the associated linear vertical betatron motion. In Fig. 2 exactly the same set of parameters has been used but now also the excitations due to longitudinal wakes are taken into account. We see that the islands widths have strongly increased which indicates a strong contribution from the impedance-caused wakefields.

For the case of no wakefields the agreement of our results with the ones of a simulation program written by one of us (T. Suzuki) [7] is excellent. No simulation for the combined wakefield-dispersion effect has been performed by us so far. This will be done in some later work. As a conclusion we may state that it is possible to describe the effects of coupled synchro-betatron motion in an efficient way using well known theories of nonlinear differential equations and perturbation techniques. Extension to even more effects like transverse wakefields and even sextupole magnets seems basically possible and will be considered in future papers by us.

References

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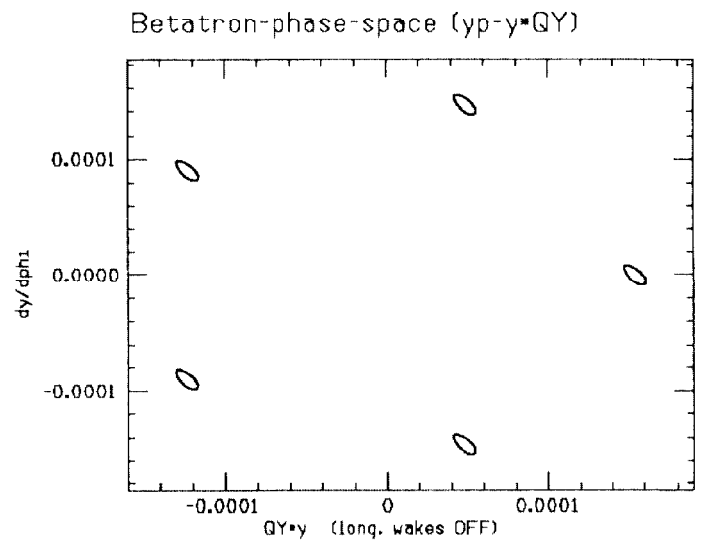


Fig. 1 - Betatron phase space (y, y') with dispersion in cavities.

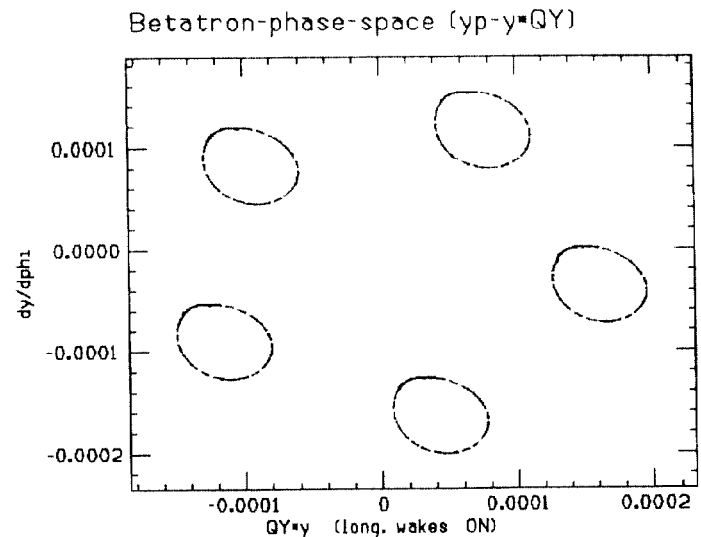


Fig. 2 - Betatron phase space (y, y') with dispersion in cavities and longitudinal wakefields.