ANALYTIC CALCULATION OF THE DYNAMICAL APERTURE FOR THE TWO DIMENSIONAL BETATRON MOTION IN STORAGE RINGS

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1. Introduction

In this contribution we study the on-momentum nonlinear equations of motion for the coupled transverse motion of a single charged particle in a storage ring. We seek for the maximum initial linear amplitudes in the two transverse directions x and y which lead to bounded particle motion as t tends to infinity. Although we restrict ourselves to sextupole fields in this paper, we may easily extend the method to any order multipole.

The aim of this work is to derive an analytic approximate expression for the dynamical aperture. We approach the solutions for x and y by use of a classical secular perturbation theory [1], [2]. Every coefficient of the perturbation series can be expressed as an analytic function of all the lower order coefficients. Although perturbation theory if it is evaluated to certain specific order leads only to an approximation in terms of bounded (trigonometric) functions we may derive information about the stability limit by considering the convergency radius of the general perturbation series. This is done in the present paper by deriving an aproximate analytic expression for the n-th order perturbation contribution of the whole series using only results up to second order. The actual calculations have been performed for the fully two dimensional case but for simplicity we shall explain only the one dimensional case of the pure horizontal motion. All the details for two dimensional motion can be found in [3] and [4].

2. The equations of motion

The nonlinear equations of motion can be found at many places in the literature (eg. [3], [4], [5], [6]):
:

$$\frac{d^2x}{ds^2} - K(s)\ x - \frac{1}{2} K'(s)\ (x^2 - y^2) = 0 \qquad (1)$$

$$\frac{d^2x}{ds^2} + K(s)\ y + K'(s)\ xy = 0 \qquad (2)$$

After transformation to Courant and Snyder variables [7] :

$$\theta_x = \frac{ds}{Q_x\beta_x} \quad , \quad \theta_x = \frac{ds}{Q_y\beta_y} \qquad (3)$$

the linear parts of these equations become of the linear harmonic oscillator type :

$$\frac{d^2x}{d\theta^2{}_x} + Q^2{}_x x - \frac{1}{2} K'(s)\ \beta_x^{3/2}(s)\ Q^2{}_x[\beta_x(s)x^2 - \beta^2{}_y(s)y^2] = 0 \qquad (4)$$

$$\frac{d^2y}{d\theta^2{}_y} + Q_y^2 y + K'(s)\ Q^2{}_y\beta^2{}_y(s)\ \beta_x^{1/2}(s)\ xy = 0 \qquad (5)$$

We choose the initial conditions :

$$x(0) = A,\ dx/d\theta_x(0) = 0,\ y(0) = B,\ dy/d\theta_y(0) = 0 \qquad (6)$$

Now we ask for the maximum values of A and B which lead to bounded motion if θ_x and θ_y tend to infinity.

3. Secular perturbation theory

For this part, we restrict ourselves to the case of the pure horizontal motion $x(\theta_x)$. The basic method remains the same for coupled motion in the x, y-plane. Detailed information about this case is contained in [3] and [4]. The equation for the horizontal motion is :

$$\frac{d^2x}{d\theta^2} + Q^2{}_0 x + \varepsilon f(\theta)\ x^2 = 0,\ f(\theta) = -\frac{1}{2} K'(\theta)\ Q^2\ \beta^{5/2}(\theta) \qquad (7)$$

The basic idea of secular perturbation theory [1], [2] is to expand x as well as the total tune Q as power series in the control parameter ε :

$$x = x_0 + \varepsilon\ x_1 + \varepsilon^2\ x_2 + \ldots + \varepsilon^n\ x_n + \ldots \qquad (8)$$

$$\omega = 1 + \varepsilon\ \omega_1 + \varepsilon^2\ \omega_2 + \ldots + \varepsilon^n\ \omega_n + \ldots \qquad (9)$$

$$Q = Q_0\omega \qquad (10)$$

If we transform the independent variable θ according to $\bar\theta = \theta\omega$ we find :

$$\omega^2\ \ddot{x} + Q^2{}_0\ x + \varepsilon\ f(\bar\theta)\ x^2 = 0 \qquad (11)$$

If we insert the perturbation expansion [8] into [7] we find a general expression for the n-th order perturbation contribution :

$$\ddot{x}_n + Q^2{}_0\ x_n = -f(\bar\theta) \sum_{j=1}^{n-1} x_j\ x_{n-1-j} - \sum_{j=0}^{n-1} \ddot{x}_j\ \Omega_{n-j} \qquad (12)$$

The Ω_n are defined as :

$$\Omega_n = \sum_{i=0}^{n} \omega_i\ \omega_{n-i} \qquad (13)$$

As we can see the n equations for x_1 to x_n contain n free parameters ω_i for the nonlinear frequency correction. According to [3] and [4], these corrections are chosen such as to cancel secular terms in any of the n equations for the perturbing contributions. Secular terms are defined as trigonometric contributions to the right hand side of (12) proportional to cos $Q\theta$ and sin $Q\theta$. This would result in a linearily increasing solution of (11) leading to a non uniformly valid perturbation representation of $x(\theta)$. The solutions x_n can be found by recursively solving the system (11) and up to 2nd order are given in [4] :

$$x_0(\bar\theta) = A \cos Q\bar\theta \qquad (14)$$

$$x_1(\bar\theta) = A^2\ q_1(\bar\theta) - A^2\ q_1(0) \cos Q\bar\theta$$

with

$$q_1(\bar\theta) = -\frac{1}{4} \sum_{n=0}^{\infty} a_n \left[\frac{2\cos n\bar\theta}{Q^2 - n^2} - \frac{\cos(2Q\pm n)\bar\theta}{(3Q\pm n)(Q\pm n)} \right] \qquad (16)$$

$$x_2(\bar\theta) = A^3\ q_2(\bar\theta) - A^3\ q_2(0) \cos Q\bar\theta \qquad (17)$$

with

$$q_2(\bar{\theta}) = \frac{1}{8} \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} a_m a_n \left[\frac{2\cos[(m\pm n\pm Q)\bar{\theta}]}{(Q^2-n^2)[Q^2-(m\pm n\pm Q)^2]} - \frac{\cos[(pQ\pm n\pm m)\bar{\theta}]}{(3Q\pm n)(Q\pm n)[Q^2-(pQ\pm n\pm m)^2]} \right] \quad (18)$$

$$p = 1,3$$

The a_n are the Fourier coefficients of $f(\bar{\theta})$ in Eq. (7). We assumed $f(\theta)$ to be a symmetric function with respect to $\theta = 0$. The frequency correction ω_2 is given in [4]. The first order contribution to the nonlinear detuning vanishes. Clearly, we observe the occurence of integer and third integer resonance denominators in x_1 and also fourth integer resonances in x_2.

4. Convergency radius of the perturbation series and dynaminal aperture

The perturbation results describing the actual solution $x(\theta)$ to a certain approximation do not directly tell us something about the limit to unbounded motion since to any order we obtain contributions in terms of bounded (trigonometric) functions (at least off-resonances, i.e. for any irrational tune). On the other hand, we may deal not only with perturbation results to a specific order, but with the perturbation series as a whole. It has been the idea of one of us (H. Moshammer, [4]) to derive an approximate expression for the n-th order perturbation contribution as an analytic function of n. Then all coefficients of the series (8) are given analytically and any formula for the convergency radius of a Taylor series can be used. Using Eq. (12) for the n-th order contribution $x_n(\theta)$ we may prove that x_n may be represented as :

$$x_n = - x_n^P(0)\cos(Q\tilde{\theta}) + x_n^P(\tilde{\theta}) \quad (19)$$

with

$$x_n^P(\tilde{\theta}) = \sum_{\ell=1}^{n} c_\ell^n q_\ell(\tilde{\theta}) \quad (20)$$

where the q_n are solutions of the following equations :

$$\ddot{q}_n + Q^2 q_n = -f(\tilde{\theta}) \sum_{j=0}^{n-1} q_j q_{n-1-j} - \frac{1}{A^{n+1}} \sum_{j=0}^{n-1} \ddot{x}_j \Omega_{n-j} \quad (21)$$

The coefficients $c^n{}_\ell$ in (20) are given by :

$$c_\ell^n = A^{n+1} \sum_{m=0}^{n-1} (-1)^m \frac{(n+m)!}{(n+1)!} \times \frac{\ell+1}{k_1!k_2!\ldots k_{n-1}!} \prod_{j=1}^{n=1} q_j(0) \quad (22)$$

where :

$$n = \ell + \sum_{j=1}^{n-1} jk_j \quad ; \quad m = \sum_{j=1}^{n-1} k_j \quad (23)$$

With (8), (19) and (20) we then find the formally exact solution for $x(\tilde{\theta})$:

$$x(\tilde{\theta}) = A\cos(Q\tilde{\theta}) + \sum_{\ell=0}^{\infty} \left(q_\ell(\theta) - \cos(Q\tilde{\theta})\, q_\ell(0)\right) \sum_{n=1}^{\infty} c_\ell^n \quad (24)$$

A necessary (possibly not sufficient) condition for (24) to converge is the convergency of all the sums :

$$S_n = \sum_{n=1}^{\infty} c_\ell^n \quad (25)$$

We use d'Alembert criterion for convergency of an infinite series which tells us that (25) converges if :

$$\lim_{n\to\infty} \left| \frac{c_\ell^{n+1}}{c_\ell{}^n} \right| < 1 \quad (26)$$

5. Restriction to second order perturbation theory

The coefficients $c^n{}_\ell$ in the analytic expression for $x(\tilde{\theta})$ (Eq. (24)) explicitly contain the pertubation contributions q_n being defined by (21): The explicit analytic solutions for $q_1(\tilde{\theta})$ and $q_2(\theta)$ are given by Eqs. (16) and (18). The idea is to keep only q_1 and q_2 in (22) and to neglect all higher order functions q_ℓ where $\ell > 2$. This approximation still keeps the n-dependence of $c^n{}_\ell$ in (22) so that d'Alemberts criterion (25) can be used to determine the approximate convergency-limit of the series (24). Since the coefficients $c^n{}_\ell$ contain the oscillation amplitude A (Eq. (21)) the so-obtained criterion at the same time results in an expression for the limiting stable amplitude. Using (22) together with (26) we find a result independent of ℓ in the following form :

$$A_{lim} = \left| \frac{1}{q_1(0)\sqrt{G(\lambda)}} \right| \quad ; \quad \lambda = - \frac{q_2(0)}{q_1(0)^2} \quad (27)$$

with

$$G(\lambda) = \left| \lim_{\ell\to\infty} \frac{1 + \sum_{k=0}^{\ell+1} \frac{(3\ell+k+4)!\,(2\ell+2)!}{(\ell+1-k)!(4\ell+5)!(2k)!} \lambda^{-k}}{(2\ell+3)(2\ell+4) \sum_{k=0}^{\ell+1} \frac{(3\ell+1+k)!(2\ell+2)!}{(\ell-k)!(4\ell+5)!(2k)!} \lambda^{-k}} \right| \quad (28)$$

Unfortunaly $G(\lambda)$ cannot be evaluated in closed form. However, we may prove [3] that G has a simple asymptotic behaviour :

$$G(\lambda) \sim \frac{27}{9} \lambda \quad ; \quad \lambda \to \infty \quad (29)$$

For $\lambda = 0$ we obtain $G(0) = 16$. From (16, (18), (27) and (28) we see that the leading effects of the third and fourth integer resonances are contained in this theory.

6. Two-dimensional motion

We now ask for the limiting amplitudes A and B in x- and y-direction according to Eqs. (4), (5) and (6). The method used is precisely the same as the one explained in Chapt. 3, 4 and 5 and we just write down the final results for A_{lim} and B_{lim}. All the details of the derivations can be found in [3] and [4]. The results again up to second order are given by :

$$A_{lim} = \mathrm{Min}\,[A_1, A_2, A_3, A_4] \quad (30)$$

$$B_{lim} = \mathrm{Min}\,[B_1, B_2, B_3, B_4] \quad (31)$$

with

$$A_1 = |4q_{11}(0)|^{-1},\ A_2 = 2|27q_{21}(0)|^{-1/2},$$

$$A_3 = |p_{11}(0)|^{-1},\ A_4 = |p_{21}(0)|^{-1/2} \quad (32)$$

$$B_1 = |q_{22}(0)|^{-1/2}, \qquad B_2 = 2|27p_{22}(0)|^{-1/2},$$

$$B_3 = 2|27p_{11}(0)q_{12}(0)|^{-1/2}, \; B_4 = |4q_{11}(0)q_{12}(0)|^{-1/2} \quad (33)$$

where q_{11}, q_{12}, q_{21}, q_{22}, p_{11}, p_{21} and p_{22} are solutions of the following linear second order differential equations :

$$\frac{d^2q_{11}}{d\theta^2_x} + Q_x^2 q_{11} = \frac{1}{2} Q_x^2 \, k'(s)\beta_x^{5/2}(s)\cos^2(Q_x\theta_x) \quad (34)$$

$$\frac{d^2q_{12}}{d\theta^2_x} + Q_x^2 q_{11} = -\frac{1}{2} Q_x^2 \, k'(s)\beta_x^{3/2}(s)\beta_y(s)\cos^2(Q_y\theta_y) \quad (35)$$

$$\frac{d^2q_{21}}{d\theta^2_x} + Q_x^2 q_{21} = Q_x^2 \, k'(s)\beta_x^{5/2}(s)q_{11}(\theta_x)\cos^2(Q_x\theta_x) \quad (36)$$

$$\frac{d^2q_{22}}{d\theta^2_x} + Q_x^2 q_{22} = -Q_x^2 k'(s)\beta_x^{3/2}[\beta_x(s)q_{12}(\theta_x)\cos(Q_x\theta_x) +$$

$$+ \beta_y(s)p_{11}(\theta_y)\cos(Q_y\theta_y)] \quad (37)$$

$$\frac{d^2p_{11}}{d\theta^2_y} + Q_y^2 p_{11} = -Q_y^2 k'(s)\beta_y^2(s)\beta_x^{1/2}(s)\cos(Q_x\theta_x)\cos(Q_y\theta_y) \quad (38)$$

$$\frac{d^2p_{21}}{d\theta^2_y} + Q_y^2 p_{21} = -Q_y^2 k'(s)\beta_y^2(s)\beta_x^{1/2}(s)[q_{11}(\theta_x)\cos(Q_y\theta_y) +$$

$$+ p_{11}(\theta_y)\cos(Q_x\theta_x)] \quad (39)$$

$$\frac{d^2p_{22}}{d\theta^2_y} + Q_y^2 p_{22} = -Q_y^2 k'(s)\beta_y^2(s)\beta_x^{1/2}(s)q_{12}(\theta_x)\cos(Q_y\theta_y) \quad (40)$$

7. Application of the results and discussion

We applied our theory to the existing model of the LEP machine containing all setupoles and all the experimental insertions. In Fig. 1 we vary the horizontal tune Q_x for a superperiod (one quarter of the whole machine) from 17.25 to 17.75 and calculate the maximum initial amplitude A_{lim} on the y-axis $[x(0) = A, \dot{x}(0) = 0, y(0) = 0, \dot{y}(0) = 0]$ at the high-β interaction point. The stars correspond to the analytic method while the crosses are the tracking results for 400 turns. Except in the interval $Q_x = 17.35$ to $Q_x = 17.45$ the analytic results agree reasonably well with the tracking (kick-code). Clearly we see the effect of the third integer resonance ($Q_x = 17.333...$) and half integer as well as fourth integer resonances at $Q_x = 17.5$, $Q_x = 17.25$ and $Q_x = 17.75$ respectively. The dip at $Q_x = 17.6$ due to a fifth order resonance (3:5) indicated by the tracking program is not covered by second order perturbation theory as can be seen from Eqs. (16) and (18). In Fig. 2 we start the motion on the y-axis $[x(0) = 0, \dot{x}(0) = 0, y(0) = B, \dot{y}(0) = 0]$ and look for B_{lim} as function of the strength of the sextupole family SF1. Again the agreement with tracking is relatively good although a certain "overshooting" of the analytic results with respect to tracking is observed. Probably an extension of the perturbation theory to third order would further improve the results.

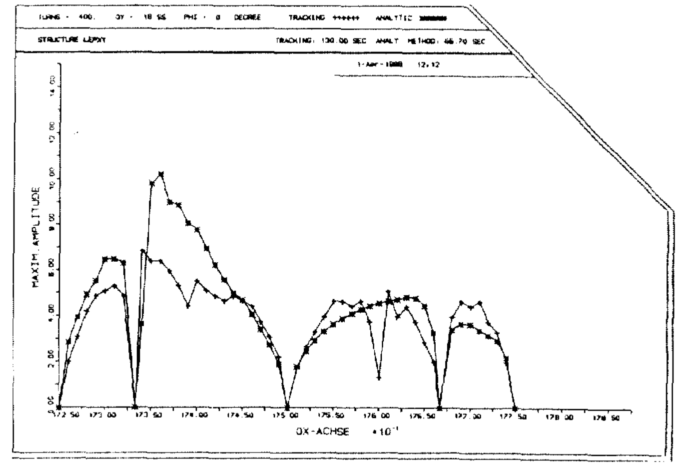

Fig. 1 - Maximum stable initial amplitude on the x-axis for LEP taken at the low-β interaction point as a function of the horizontal tune Q_x for a superperiod.

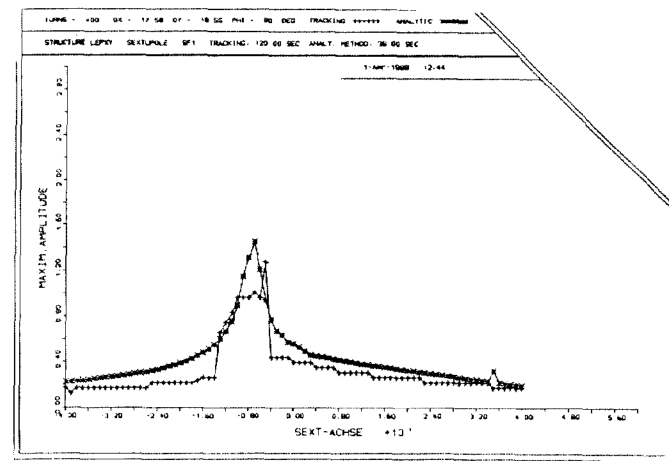

Fig. 2 - Maximum stable initial amplitude on the y-axis for LEP taken at the low-β interaction point as a function of the strength of the sextupole family SF1.

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