# PULSARS AS COSMIC RAY ACCELERATORS - THE CRITICAL FREQUENCY

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## Abstract

The question is discussed under which conditions a strongly magnetized star rotating with its vector of angular velocity $\underline{\omega}$ perpendicular to its vector of magnetic dipole moment $\underline{\mu}$ might be able to evacuate a certain spatial "region of acceleration" from electrically charged particles such as electrons and ions. A preliminary estimate based on the test-particle approach suggests that for this to happen the angular velocity $\omega$ has to exceed a "critical value" $\omega_c$ which is proportional to $\mu^{-1/2}$.

## 1. Objective of this note

In this paper I will consider the possibility that an "orthogonal rotator", i.e. a rapidly rotating, strongly magnetized star of mass M rotating with its vector of angular velocity $\underline{\omega}$ perpendicular to its vector of magnetic dipole moment $\underline{\mu}$ may be able to evacuate a certain spatial "region of acceleration" in its surroundings from electrically charged particles. The existence of such a "region of acceleration" is a necessary prerequisite for the functioning of pulsars as cosmic ray accelerators (Thielheim, 1986).

## 2. Basic assumptions and approximative procedures

For this purpose I will perform an estimate on the balancing of gravitation and acceleration reaction in the vacuum wave field of a rotating dipole.

It is appropriate to point out that if in reality such an evacuated spatial "region of acceleration" does exist in the vicinity of a pulsar the electromagnetic field inside this region may well differ from the vacuum field, due to contributions produced by collective particle motion.

Furthermore, for reasons of simplicity, my discussion will be restricted to places near the axis of rotation, where the electromagnetic wave field locally can be approximated by a plane circularly polarized wave

$$\underline{E} = E_o \cdot \left\{ \underline{Y}_o \sin\Psi + \underline{Z}_o \cos\Psi \right\} \tag{1}$$

$$\underline{H} = H_o \cdot \left\{ -\underline{Y}_o \cos\Psi + \underline{Z}_o \sin\Psi \right\} \tag{2}$$

$\underline{y}_o$ is the unit vector parallel to the axis of rotation.
$\underline{x}_o$ is the unit vector describing the direction of wave propagation, which according to what has been said before, is perpendicular to $\underline{y}_o$. $\underline{z}_o$ is the unit vector perpendicular to both $\underline{x}_o$ and $\underline{y}_o$, such that these three unit vectors constitute a right hand rectangular tripod.

$$\Psi = \omega \left\{ t - x/c \right\} \tag{3}$$

is the phase, and

$$E_o = H_o = \mu / r_L^2 r_o \tag{4}$$

is the wave amplitude determined by the radial coordinate $r_o$ of the test-particle under consideration.

## 3. Parameters involvec

The rotational state of the magnetized star is characterized by the parameter

$$r_L = c/\omega \tag{5}$$

which is the so-called "light radius", more precisely: the radius of the light cylinder. Also the following parameter will be used

$$\omega_L = e H_o / mc \,, \tag{6}$$

where e is the electric charge and m the mass of the test-particle.

$$r_T = \left( e\mu / mc^2 \right)^{1/2} \tag{7}$$

is the "typical radius" representing essentially the electromagnetic properties of the system at hand. The two parameters are related through

$$\omega_L / \omega = r_T^2 / r_L r_o \,. \tag{8}$$

The "plane wave approximation" is justified as long as $r_o$ is large compared with the "acceleration boundary (Thielheim, 1987)

$$r_B = \left( r_T / r_L \right)^{1/3} r_T \,. \tag{9}$$

$$L_o = c\,\tau_o = 2\,e^2 / 3\,mc^2 \tag{10}$$

is the "radiation reaction length". The gravitational acceleration of the test-particle is

$$g = \Gamma M / r_o^2 \tag{11}$$

where M is the mass of the magnetized star and $\Gamma$ is the gravitational constant. Also use will be made of the "gravitational radius"

$$r_G = \Gamma M / c^2 \qquad (12)$$

The following notation is used: $u_x = dx/cd\tau = \dot{x}/c$ etc., $u_o = dx_o/cd\tau = \dot{x}_o/c$, $dx_o = cdt$, $dt = \gamma d\tau$, $v_x = dx/dt$. etc., $\gamma = (1 - v_x^2/c^2 - v_y^2/c^2 - v_z^2/c^2)^{-1/2}$ and $u^2 = u_o^2 - u_x^2 - u_y^2 - u_z^2$.

## 4. Equations of motion

An electrically charged test-particle, e.g. an electron, finds itself subject to the influence of the Lorentz force of a plane electromagnetic wave, which, for example, near to the axis of rotation, is a circularly polarized plane wave and subject to the gravitational force and the force of electromagnetic radiation reaction. The appropriate equation of motion is the Lorentz-Dirac equation with additional terms providing for the gravitational force

$$\dot{u}_x = \omega_L \{u_y \sin\psi + u_z \cos\psi\} - g u_o/c + \tau_o[\ddot{u}_x - \dot{u}^2 u_x] \qquad (13a)$$

$$\dot{u}_y = \omega_L (\dot{\psi}/c) \sin\psi \qquad + \tau_o[\ddot{u}_y - \dot{u}^2 u_y] \qquad (13b)$$

$$\dot{u}_z = \omega_L (\dot{\psi}/c) \cos\psi \qquad + \tau_o[\ddot{u}_z - \dot{u}^2 u_z] \qquad (13c)$$

$$\dot{u}_o = \omega_L \{u_y \sin\psi + u_z \cos\psi\} - g u_x/c + \tau_o[\ddot{u}_o - \dot{u}^2 u_o] \qquad (13d)$$

In the Lorentz approximation the equations of motion (13a) - (13d) reduce to

$$\dot{u}_x = \omega_L \{u_y \sin\psi + u_z \cos\psi\} \qquad (14a)$$

$$\dot{u}_y = \omega_L (\dot{\psi}/\omega) \sin\psi \qquad (14b)$$

$$\dot{u}_z = \omega_L (\dot{\psi}/\omega) \cos\psi \qquad (14c)$$

$$\dot{u}_o = \omega_L \{u_y \sin\psi + u_z \cos\psi\} \qquad (14d)$$

leading among other consequences to

$$u_o^2 = -(\omega_L/\omega)^2 \dot{\psi}^2 \qquad (15)$$

$$\ddot{u}_o = \ddot{u}_x \qquad (16)$$

If the Lorentz approximation for $\dot{u}^2$, $\ddot{u}_o$, $\ddot{u}_x$ and $\ddot{u}_y$ is introduced into those terms of the equations of motion (13a-13d), which describe radiation reaction, one arrives at the Landau approximation of which I only need

$$\dot{u}_x = \omega_L \{u_y \sin\psi + u_z \cos\psi\} - g u_o/c + \tau_o\left[\ddot{u}_x - \left(\frac{\omega_L}{\omega}\right)^2 \dot{\psi}^2 u_x\right] \qquad (17a)$$

$$\dot{u}_o = \omega_L \{u_y \sin\psi + u_z \cos\psi\} - g u_x/c + \tau_o\left[\ddot{u}_o - \left(\frac{\omega_L}{\omega}\right)^2 \dot{\psi}^2 u_o\right] \qquad (17b)$$

Subtracting these two equations of motion from each other leads to a non-linear differential equation for the phase $\psi$ as a function of the eigentime $\tau$ (where again the point denotes differentiation with respect to the eigentime $\tau$ ):

$$\ddot{\psi} = g\dot{\psi}/\tau - \tau_o (\omega_L/\omega)^2 \dot{\psi}^3 \qquad (18)$$

From this, a corresponding non-linear differential equation is obtained for the eigenfrequency $\eta = \dot{\psi}$ as a function of the phase $\psi$

$$d\eta/d\psi = g/c - \tau_o (\omega_L/\omega)^2 \eta^2 \qquad (19)$$

## 5. Discussion of Limiting Cases

Before considering the question which was formulated at the beginning of this note it is useful to discuss two limiting cases: The first case is the one with no radiation reaction:

$$d\eta / d\psi = g/c \qquad (20)$$

i.e.

$$\eta - \eta_o = (g/c)\cdot(\psi - \psi_o) \qquad (21)$$

leading to an exponential increase of the eigenfrequency $\eta$ with increasing eigentime $\tau$:

$$\eta = (g\psi_o/c)\cdot \exp(g\tau/c) \quad . \qquad (22)$$

Since the eigentime $\tau$ increases monotonously when the time coordinate t does, the test-particle in this case experiences a wave field, which is increasingly blue shifted. Thus, the test-particle is globally accelerated upstream falling onto the magnetized star.

Alternatively, the second case is the one with no gravitational force:

$$d\eta/d\psi = -\tau_o (\omega_L/\omega)^2 \eta^2 \qquad (23)$$

i.e.

$$\psi - \psi_o = (1/\tau_o)\cdot(\omega/\omega_L)^2\cdot(1/\eta - 1/\eta_o) \qquad (24)$$

leading asymptotically to a monotonous decrease of the eigenfrequency $\eta$ with increasing eigentime $\tau$ :

$$\eta \rightarrow (\omega/\omega_L)\cdot(\tau_o \tau)^{-1/2} \qquad (25)$$

Obviously the test-particle in this case experiences a wave field which increasingly is red shifted. Thus the test-particle is ultimately accelerated downstream receding from the magnetized star. This result is especially interesting because it demonstrates that it is with the help of radiation reaction that an electrically charged mass point can be globally accelerated by a plane wave field though it is well known that there is no such global acceleration without radiation reaction!

## 6. Balancing Gravitation and Radiation Reaction

The differential equation (19) which determines the eigenfrequency (i.e. the frequency of the wave as seen from an inertial frame of reference in which the test-particle is momentarily at rest) is easily solved

$$\eta - \eta_o = \hat{\eta} \cdot \{ tgh(\gamma/\hat{\gamma}) - tgh(\gamma_o/\hat{\gamma}) \} \tag{26}$$

where

$$\hat{\eta} = (\omega/\omega_L) \cdot \sqrt{g/c\tau_o} \tag{27}$$

and

$$\hat{\gamma} = (\omega/\omega_L) \cdot \sqrt{c/g\tau_o} \tag{28}$$

$\eta_o$ and $\gamma_o$ are chosen by initial conditions such that

$$\eta_o = \hat{\eta}\ tgh(\gamma_o/\hat{\gamma}) \tag{29}$$

i.e.

$$\eta = \hat{\eta}\ tgh(\gamma/\hat{\gamma}) \tag{30}$$

Asymptotically, for $\gamma \rightarrow \infty$, one has

$$\eta \rightarrow \hat{\eta} = const. \tag{31}$$

The suggestion of the present estimate is that while the test-particle moves under the simultaneous influence of a very strong force (the Lorentz force) and two very weak forces (the gravitational force and the radiation reaction force, respectively) the latter two forces have to compensate in the mean, while the test-particle performs a great number of loops (Thielheim, 1988). (A transverse drift motion superimposed on this quasi periodic motion is neglected as well as effects produced by the spherical structure of the wave field).

Obviously, there is no global acceleration towards the magnetized star or away from it for

$$\omega = \hat{\eta} \tag{32}$$

which, by (4), (5), (6), (7), (11) and (12) is equivalent to

$$\omega = (c/r_T) \cdot (r_G/L_o)^{1/4} \tag{33}$$

This estimate is for just one single electron as a test-particle. Still one has to notice, that although it is the electron on which radiation reaction is most effective it is but the proton on which the gravitational force is the stronger. Bearing in mind that the electron und the proton are coupled together by the need to keep the plasma quasi neutral, the "critical frequency" $\omega_c$ of rotation of the magnetized star, for which gravitation and radiation reaction are expected to compensate in the mean, may be estimated after replacing $r_G$ by $R_G = m_p r_G/m$, where $m_p$ is the mass of the proton, leading to

$$\omega_c = (c/r_T) \cdot (R_G/L_o)^{1/4} \tag{34}$$

The numerical value for this conservative estimate for a magnetized neutron star with the magnetic dipole moment $\mu = 10^{30}$ G cm³ is about 40 Hz.

## References


Thielheim, K.O., Proc. 2nd ESO/CERN Conf. München, 1986, pp 317-320

Thielheim, K.O., Journal of Physics 20, L203-L206, 1987

Thielheim, K.O., Proc. of the Advanced CERN Accelerator School, Berlin, 1988