CHROMATICITY CONTROL IN NONSCALING FFAGS BY SEXTAPOLES

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Abstract

Because of their high repetition rate and large apertures, FFAGs are proposed for high-current medical accelerators suitable for cancer therapy. The linear-field nonscaling FFAG is made from repeating cells containing D and F combined function magnets. The F and D elements are horizontally focusing and defocusing, respectively. The betatron tune profiles decrease with momentum; this leads to the crossing of resonances, possibly leading to emittance increase. We examine how sextapole magnets may be used to flatten the tune profile; in particular (i) whether it is better to place them at the D or F; (ii) what strength is required; and (iii) what is their effect on the closed orbits and path length? Chromaticity is corrected by coupling focusing strength to dispersion, which is far stronger in the F element. The zeros of the orbit dispersion become the poles of the "sextapole strength to flatten the tune at some particular momentum". Consequently, a weak F sextapole can produce a substantial horizontal tune flattening, and has little impact on other optical properties. Contrarily, placing the necessarily strong sextapole at the D element may destroy the dynamic aperture.

INTRODUCTION

To find the closed orbits and path length we follow the same *kick model* of the combined-function magnet elements as reported in Refs. [3, 1]. The same notations are also adhered to. Throughout, we adopt physical units wherein the particle charge and speed of light are unity. To find the tunes we employ power series expansions in the quadrupole strength as was reported in Ref. [2]. Here we consider a degenerate F0D0 lattice with equal element lengths and gradients. Triplet and doublet cells, and split quadrupole strengths, are studies in the extensive Ref. [4], from which the brief results reported here are drawn. Here we study the case of bending at momentum p_c purely in the D, but the general case of bending shared between D and F is treated Ref. [4].

We shall consider the case of weak sexapoles. The motivation for this is two-fold: (a) strong sextapoles will reduce the dynamic aperture, and (b) weak sextapoles lend themself to a simple pertrubation treatment. The weak sextapole condition is $2|\sigma|\theta l_0 \ll |\beta|$. $2l_0\theta$ is roughly the aperture of the magnet. The condition is essentially that the product of integrated sextapole strength and aperture be much less than the integrated quadrupole strength. For simplicity, we consider a single sextapole per cell placed at D or F elemenst, but not both.

* TRIUMF receives federal funding via a contribution agreement through the National Research Council of Canada.

F SEXTAPOLE

The bend angle of a half cell is θ; this is the bend in the D element at the reference momentum p_c. Let primes denote derivatives with respect to transverse displacements; the quadrupole gradient is B', and similarly for the sextapole B''. The integrated strength (gradient×length) of the half quadrupole elements is $\beta = B' \times l$, and their separations are l_0. We consider a degenerate F0D0 lattice with equal half quadrupole strengths $\beta_d = \beta_f = \beta$. We take also the element half lengths to be equal, $l_d = l_f = l$. However, we add a sextapole to be superposed at the F element. The integrated strength of the half sextapole element is $\sigma_f = B'' \times l$. The quadrupole and sextapole centres are coincident. For brevity, we shall denote $l_0\beta \equiv p_0$.

The angular deflections at D and F are, respectively:

$$\begin{aligned}(\psi_{d1} + \psi_{d2}) &= 2[p_c\theta - \beta x_d]/p && (1)\\ (\psi_{f1} + \psi_{f2}) &= 2[\beta x_f + \sigma_f x_f^2]/p\,. && (2)\end{aligned}$$

The closed-orbit displacements versus momentum are:

$$\begin{aligned}x_d &= \frac{(p-p_c)(p-p_0)}{\beta p_0}\theta - \sigma_f\frac{[(p-p_c)p]^2 p}{(\beta p_0)^3}\theta^2 && (3)\\ x_f &= \frac{(p-p_c)p}{\beta p_0}\theta - \sigma_f\frac{[(p-p_c)p]^2(p+p_0)}{(\beta p_0)^3}\theta^2\,. && (4)\end{aligned}$$

These expressions are not limited to weak sextapoles; they are exact to order θ^2. The introduction of the sextapole raises the momentum dependence from quadratic to quintic. The path length to order θ^3 is:

$$\begin{aligned}\mathcal{L}(p) &= 2l_0 + (p-p_c)(3p-p_c-2p_0)\theta^2/(\beta p_0)\\ &- 2\sigma_f[(p-p_c)p]^2(2p-p_c)\theta^3/(\beta p_0)^3\,. && (5)\end{aligned}$$

Let the momenta $\check{p}$ and $\hat{p}$ denote the lower and upper extent of the machine operating range. The basic lattice is designed in the absence of the sextapole. The focusing strength is constrained by the condition $p_0 \equiv l_0\beta \leq \check{p}$, and the path length is usually chosen to satisfy $\mathcal{L}(\check{p}) = \mathcal{L}(\hat{p})$.

Incremental Kicks and Focusing

We substitute $x_d(p) + x$ and $x_f(p) + x$ from (3,4) into the kick equations (1,2), and recover the terms linear in x:

$$\begin{aligned}\delta\psi_d/x &= -2\beta/p && (6)\\ \delta\psi_f/x &= 2(\beta/p) + 4(p-p_c)/(\beta p_0)(\sigma_f\theta)\\ &- 4(p-p_c)^2 p(p+p_0)/(\beta p_0)^3(\sigma_f\theta)^2\,. && (7)\end{aligned}$$

The analogous deflections in the vertical plane are:

$$\delta\psi_d/y = +2\beta/p \qquad \delta\psi_f/y = -y[\beta + 2\sigma_f x_f]/p\,, \quad (8)$$

Notice, that the subscripts d, f here refer to the same elements as they do in the horizontal plane; e.g. $(\delta\psi_f)/y$ is vertically defocusing. The incremental angular deflections result in transverse focusing.

Betatron Tunes

Because of the feed-down focusing from the sextapole, the betatron tunes are split. For brevity, let $S_f = 2\sigma_f\theta/(l_0\beta^2)$. To first order in S_f and the element half length (l), and to second order in the element separation (l_0), the horizontal phase advance (ϕ_x) is given by:

$$\cos\phi_x = 1 - \frac{2p_0^2}{p^2} - \frac{2S_f l_0}{p}(p-p_c)(p+p_0) + \ldots \quad (9)$$
$$+l\frac{4}{3}\left[-\frac{4\beta p_0}{p^2} + \frac{S_f}{p^2}(p-p_c)(p_0^2 - 4p_0 p - 3p^2)\right] + \ldots$$

The vertical phase advance ϕ_y is given by $S_f \to -S_f$ and $p_0 \to -p_0$ only in the terms in S_f.

In the following, we adopt the limit of thin elements and set $l = 0$. Notice that the sextapole term has a single zero (at $p = p_c$) for ϕ_x, whereas it has an additional zero (at $p = p_0$) for ϕ_y. This distinction is crucial. While the sextapole can change the horizontal tune at the low momenta, it has almost no effect on the vertical tune around $l_0\beta \approx \check{p}$.

Horizontal Tune Profile Flattening

Flattening the tune variation versus momentum is equivalent to flattening $\cos\phi_x$, which is a much simpler function to deal with. There are a variety of choices which can be made, in order of increasing sextapole strength:

- tune locally flat at the high momentum $\hat{p}$
- tune locally flat at the reference momentum p_c
- locally flat at the mean momentum $\bar{p} \equiv (\check{p} + \hat{p})/2$
- equal at minimum ($\check{p}$) and maximum ($\hat{p}$) momentum

The first 3 conditions require the derivative $\partial\cos\phi_x/\partial p$ be zero at particular values of momentum $(\hat{p}, p_c, \bar{p})$, leading to

$$\sigma_f(p) = \beta^2 p_0^2/[p(p^2 + p_0 p_c)\theta] \; . \quad (10)$$

Note, $\sigma_f(p)$ has no poles except at $p = 0$, which is (usually) well outside the machine operation range. The fourth condition, treated in Ref.[4], requires $\cos\phi_x(\check{p}) = \cos\phi_x(\hat{p})$ and gives the *absolute minimum* tune range.

As an example, we take basic lattice parameters $l_0 = 3$ (metre), $\beta = 10.575$ (tesla), $\theta = \pi/50$ (radian), $\check{p} = 10$, $\hat{p} = 20$ $p_c = 17.745$ all momenta in (MeV/c). The longitudinal working point is $b = 0.3013$.

Figure 1 shows $\cos\phi_x(p)$ and the tune profile $\nu_x = \phi_x/(2\pi)$ for the five cases $\sigma_f = 0$, 4.24, 5.62, 8.17, 9.81 (T/m), shown red, magenta, green, cyan, blue respectively. The former three values satisfy the weak sextapole condition, but the latter are marginal.

Influence on vertical tune The cross-talk effect of the F-sextapole on the (per cell) vertical phase advance ϕ_y and tune ν_y is shown in Figure 2; there is little effect.

The slight increase of the tune at $\check{p}$ is a reason for caution in selecting σ_f and the machine base tune.

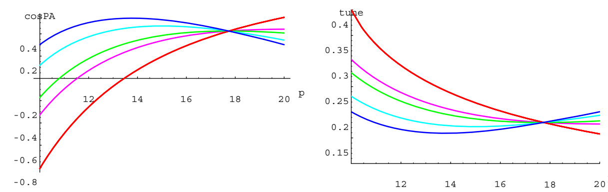

Figure 1: $\cos\phi_x$ (left) and ν_x (right) versus momentum for a variety of F-sextapole strength.

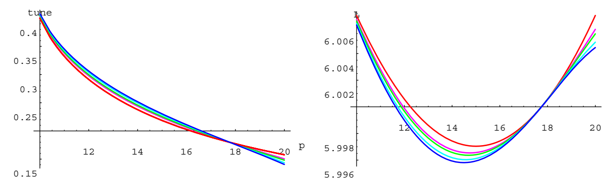

Figure 2: Tune ν_y (left) and path length (right) versus momentum for a variety of F-sextapole strength.

Closed orbits and path length The effects of the F-sextapole on the closed orbits x_d, x_f and cell path length $\mathcal{L}$ for a variety of σ_f are shown in Figs. 3 and 2 It is clear that the disturbance is a acceptable for all four values σ_f.

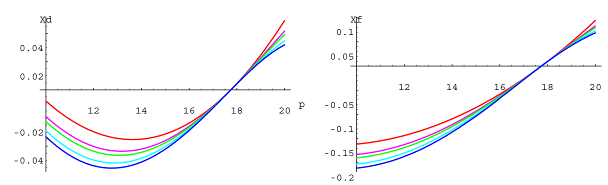

Figure 3: x_d (left) and x_f (right) versus momentum for a variety of F-sextapole strength.

D SEXTAPOLE

We repeat the exercise for the scenario that the sextapole is placed at the horizontally defocusing element. The angular deflections at D and F, respectively, are:

$$(\psi_{d1} + \psi_{d2}) = 2[p_c\theta - \beta x_d + \sigma_d x_d^2]/p \quad (11)$$
$$(\psi_{f1} + \psi_{f2}) = 2(\beta x_f)/p \; . \quad (12)$$

To first order in the sextapole strength, the off-momentum closed-orbits at D and F elements:

$$x_d = (p-p_c)(p-p_0)\theta/(\beta p_0) \quad (13)$$
$$- \sigma_d[(p-p_c)(p-p_0)]^2(p-p_0)\theta^2/(\beta p_0)^3$$
$$x_f = (p-p_c)p\theta(\beta p_0) \quad (14)$$
$$- \sigma_d[(p-p_c)(p-p_0)]^2 p\theta^2/(\beta p_0)^3 \; .$$

The expressions are not limited to weak sextapoles; they are exact to order θ^2. Compared with the case of no sextapole, the momentum dependence has been raised from quadratic to quintic. As we shall see later, the triple (in x_d) and double (in x_f) repeated zeros at $p = p_0$ will make the sextapole spectacularly *ineffective*. The path length is:

$$\begin{aligned}\mathcal{L} &= 2l_0 + (p-p_c)(3p-p_c-2p_0)\theta^2/\beta p_0 \qquad (15)\\ &+ 2\sigma_d[(p-p_c)(p-p_0)]^2(p_0+p_c-2p)\theta^3/(\beta p_0)^3\ .\end{aligned}$$

Incremental Kicks and Focusing

After making the substitutions $x_d \rightarrow x_d(p) + x$ and $x_f \rightarrow x_f(p) + x$ from (13, 14) into the kick equations (11, 12), and retaining terms linear in x and σ_d, we find

$$\delta\psi_d/x = -2\beta/p + 4(p-p_c)(p-p_0)(\sigma_d\theta)/(\beta p_0 p) \qquad (16)$$
$$\delta\psi_f/x = +2\beta/p\ . \qquad (17)$$

The angular deflections in the vertical plane are:

$$\delta\psi_d/y = +2[\beta - 2\sigma_d x_d]/p \qquad \delta\psi_f/y = -2(\beta/p)\ . \quad (18)$$

Betatron Tunes

The weak sextapole limit is the same for σ_d as it is for σ_f. Let $S_d \equiv 2\sigma_d\theta/(l_0\beta^2)$. The vertical phase advance:

$$\cos\phi_y = 1 - \frac{2p_0^2}{p^2} + \frac{2S_d l_0}{p^2}(p-p_c)(p+p_0)(p-p_0) + \quad (19)$$
$$+l\frac{4}{3}\left[-\frac{4\beta p_0}{p^2} - \frac{S_d}{p^2}(p-p_c)(p_0^2-4p_0p-3p^2)\frac{(p-p_0)}{p}\right] + \ldots$$

The horizontal ϕ_x is obtained by $S_d \rightarrow -S_d$ and $p_0 \rightarrow -p_0$ only in the S_d terms, except for the two final $(p-p_0)$.

The terms in S_d have at least two zeros, in which case the tunes are pinned at their $S_d = 0$ values at the locations $p = p_c$ and $p = l_0\beta$ irrespective of the actual value of S_d. Evidently, under that circumstance, it will be difficult to modify the tune variation unless strong sextapoles are used; and particularly for ϕ_x because of the repeated zero. Again we shall take the limit of thin elements, $l \rightarrow 0$.

Vertical Tune Profile Flattening

Consider now to flatten the tune profile by asking that the $\partial\cos\phi_y/\partial p = 0$ at some momentum. The solution is

$$\sigma_d(p) = -\beta^2 p_0^2/[(p^3 + p_0^2(p-2p_c))\theta]\ . \qquad (20)$$

σ_d has three poles, but only one is relevant; depending on parameter choices it lies between $\check{p}$ and $\bar{p}$. Equation (20) is evaluated at $\hat{p}$, p_c and $\bar{p}$ resulting in the values $\sigma_d = -7.3$, -12.1 and -31.7 (T/m) respectively. These are weaker values than for ν_y flattening with a F-sextapole, but they still violate the condition for weak sextapoles. None of them produces a significant tune flattening. Moreover, though it it not proven here, it is probable that such strong sextapoles will severly compromise the dynamic aperture for betatron oscillations about the reference orbits. The final value, which makes the tune double valued, serves only to increase the rate of tune variation with momentum.

The cases $\sigma_d = 0, -7.3, -12.1, -31.7$ are denoted by the colours red, blue, green, magenta in Fig. 4.

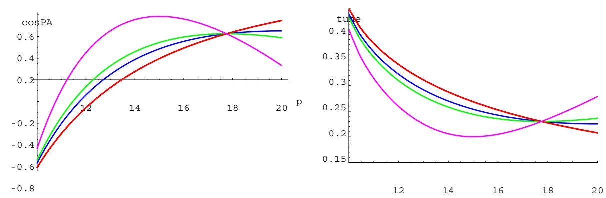

Figure 4: $\cos\phi_y$ and ν_y versus momentum for a variety of D sextapole strength σ_d.

Influence on horizontal tune The influence on ϕ_x, ν_x of σ_d chosen to locally flatten ν_y is given in Figure 5. The cross-talk is small, but the range of ν_x has increased.

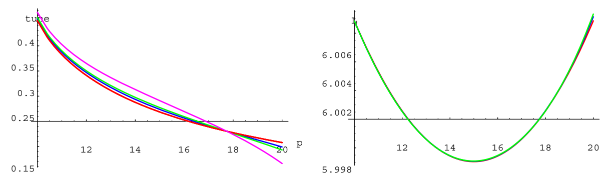

Figure 5: Tune ν_x (left) and path length (right) versus momentum for a variety of D sextapole strength σ_d.

Closed orbits and path length The effect of the vertical tune flattening on x_d, x_f and $\mathcal{L}$ is utterly negligible, as may be seen from Figure 5. However, there remains the problem that strong sextapoles are required to effect even small adjustments to the ν_y profile.

CONCLUSION

We have demonstrated that a weak F-sextapole produces a substantial flattening of the horizontal tune profile. Though a D-sextapole is more effective in flattening vertical tune, the sextapole strength for even small flattening is strong and may have severe impact on dynamic aperture.

Lattices with reverse dipole bending When the bending at p_c is shared ($\theta = \theta_d + \theta_f$) between D and F, this introduces an extra zero, at $p = -p_0\theta_f/\theta \equiv -p_f$, into the S_f part of $\cos\phi_x$. At p_f, the F-sextapole cannot change the tune. If $\theta_f < 0$, then the zero is shifted toward $p > 0$ and a stronger σ_f is required. The expressions for $\cos\phi$ with a D-sextapole differ from the $\theta_f = 0$ case by $(p-p_0) \rightarrow (p - p_0\theta_d/\theta)$ for one of the zeros which is moved slightly higher into the machine range and a stronger σ_d is required.

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