# VLASOV EQUILIBRIUM OF A PERIODICALLY TWISTED ELLIPSE-SHAPED CHARGED-PARTICLE BEAM IN A NON-AXISYMMETRIC PERIODIC MAGNETIC FOCUSING FIELD

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*Abstract*

A new Vlasov equilibrium is obtained for a periodically twisted ellipse-shaped charged-particle beam in a non-axisymmetric periodic permanent magnetic focusing field. The equilibrium distribution function is derived, and the statistical properties of the beam equilibrium are studied. The generalized envelope equations derived from the kinetic theory recovers the generalized envelope equations obtained in the cold-fluid theory when the temperature is taken to be zero. Examples of periodically twisted elliptic beam equilibrium are presented and applications are explored.

## INTRODUCTION

A fundamental understanding of the kinetic equilibrium and stability properties of high-intensity electron and ion beams in periodic focusing fields plays a central role in the design and operation of particle accelerators, such as storage rings and rf and induction linacs, as well as vacuum electron devices, such as klystrons and traveling-wave tubes with periodic permanent magnet (PPM) focusing. There are two well-known equilibria for periodically focused intense beams, namely, the Kapchinskij-Vladmirskij equilibrium [1], [2] in an alternating-gradient quadrupole magnetic focusing field and the periodically focused rigid-rotor Vlasov equilibrium [3] in a periodic solenoid magnetic focusing field. More generally, self-consistent beam distributions can be constructed with linear focusing forces as discussed in Ref. [4].

In this paper, it is shown that there exists a Vlasov equilibrium for a periodically twisted large-aspect-ratio intense charged-particle beam with a uniform density in the transverse direction propagating through a non-axisymmetric periodic magnetic focusing field.

## VLASOV EQUILIBRIUM THEORY

We consider an ellipse-shaped, continuous, intense charged-particle beam of major axis $a$ and minor axis $b$ propagating with constant axial velocity $\beta_b c\mathbf{e}_z$ through an applied non-axisymmetic periodic magnetic focusing field. The applied non-axisymmetic periodic magnetic focusing field inside a thin beam with $k_{0x}^2 a^2/6 << 1$ and $k_{0y}^2 b^2/6 << 1$ can be approximated by

$$\mathbf{B}^{\text{ext}} \cong (B_0/k_0)\left[k_{0x}^2 \cos(k_0 s)x\mathbf{e}_x + k_{0y}^2 \cos(k_0 s)y\mathbf{e}_y\right] - B_0 \sin(k_0 s)\mathbf{e}_z , \qquad (1)$$

where $k_0^2 = k_{0x}^2 + k_{0y}^2$ and $s = z$ is the axial coordinate. The associated magnetic vector potential can be expressed as $\mathbf{A}^{\text{ext}} = -B_0 \sin(k_0 s)\left(-k_{0x}^2 y\mathbf{e}_x + k_{0y}^2 x\mathbf{e}_y\right)/k_0^2$, which gives $\mathbf{B}^{\text{ext}} = \nabla\times\mathbf{A}^{\text{ext}}$.

To determine the self-electric and self-magnetic fields of the elliptic beam self-consistently, we assume that the density profile of the beam is uniform inside the beam boundary, i.e.,

$$n_b(x,y,s) = \begin{cases} N_b/\pi a(s)b(s), & \tilde{x}^2/a^2(s) + \tilde{y}^2/b^2(s) \le 1, \\ 0, & \text{otherwise.} \end{cases} \qquad (2)$$

In Eq. (2), $\tilde{x} = x\cos[\theta(s)] + y\sin[\theta(s)]$ and $\tilde{y} = -x\sin[\theta(s)] + y\cos[\theta(s)]$ are the twisted coordinates as illustrated in Fig. 1. The semi-major and semi-minor axes have the same periodicity $S = 2\pi/k_0$ as the applied magnetic field, i.e., $a(s) = a(s+S)$ and $b(s) = b(s+S)$. $N_b = \int_{-\infty}^{\infty} n_b(x,y,s)dxdy = \text{const}$ is the number of particles per unit axial length. In the paraxial approximation, the Budker parameter of the beam is assumed to be small, i.e., $q^2 N_b/mc^2 << \gamma_b$, and the transverse kinetic energy of a beam particle is assumed to be small compared with its axial kinetic energy. Here, $c$ is the speed of light in *vacuo*, $\gamma_b = \left(1-\beta_b^2\right)^{-1/2}$ is the relativistic mass factor, $q$ and $m$ are the particle charge and rest mass, respectively.

From the equilibrium Maxwell equations, the self-electric and self-magnetic fields are well known for an elliptical beam [5] with density distribution specified in

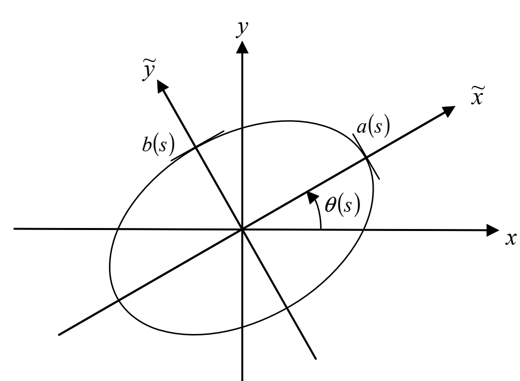


Figure 1: Laboratory and twisted coordinate systems.

Eq. (2), i.e., $\mathbf{E}^{\rm self} = -\nabla\phi^{\rm self}$ and $\mathbf{B}^{\rm self} = \nabla\times\mathbf{A}^{\rm self}$ with

$$\phi^{\rm self}(\tilde{x},\tilde{y},s) = \beta_b^{-1} A_z^{\rm self}(\tilde{x},\tilde{y},s) = -\frac{2qN_b}{a(s)+b(s)}\left[\frac{\tilde{x}^2}{a(s)}+\frac{\tilde{y}^2}{b(s)}\right] \quad (3)$$

and $\mathbf{A}^{\rm self}(\tilde{x},\tilde{y},s) = A_z^{\rm self}(\tilde{x},\tilde{y},s)\mathbf{e}_z$ for $\tilde{x}^2/a^2 + \tilde{y}^2/b^2 \le 1$.

In the paraxial approximation, the transverse motion for an individual particle in the combined self fields and applied magnetic field is described by the normalized transverse Hamiltonian $\hat{H}_\perp = H_\perp/\gamma_b\beta_b mc^2$,

$$\hat{H}_\perp(x,y,P_x,P_y,s) = \frac{1}{2}\left[P_x + \sqrt{\kappa_z(s)}\frac{k_{0y}^2}{k_0^2}y\right]^2 + \frac{1}{2}\left[P_y - \sqrt{\kappa_z(s)}\frac{k_{0x}^2}{k_0^2}x\right]^2 - \frac{K}{a(s)+b(s)}\left\{\frac{\tilde{x}^2}{a(s)}+\frac{\tilde{y}^2}{b(s)}\right\}. \quad (4)$$

In Eq. (4), $(x,P_x)$ and $(y,P_y)$ are canonical conjugate pairs, $K = 2q^2N/\gamma_b^3\beta_b^2 mc^2$ is the self-field perveance, $\sqrt{\kappa_z(s)} = qB_z(s)/2\gamma_b\beta_b mc^2$, and the normalized transverse canonical momentum $\mathbf{P}_\perp = (P_x,P_y)$ is related to the transverse mechanical momentum $\mathbf{p}_\perp$ by $\mathbf{P}_\perp = (\gamma_b\beta_b mc)^{-1}(\mathbf{p}_\perp + q\mathbf{A}_\perp^{\rm ext}/c)$.

It is convenient to transform the Hamiltonian from the Cartesian canonical variables to new canonical variables, so that the new Hamiltonian assumes a simpler form from which the invariants of the motion are easily identified. The transformation of the Hamiltonian from the Cartesian canonical variables $(x,y,P_x,P_y)$ to the new canonical variables $(x_1,y_1,P_{x1},P_{y1})$ can be obtained by successive applications of the generating functions [6]

$$F_2(x,y;\tilde{P}_x,\tilde{P}_y,s) = \tilde{P}_x\{x\cos[\theta(s)] + y\sin[\theta(s)]\} + \tilde{P}_y\{-x\sin[\theta(s)] + y\cos[\theta(s)]\}, \quad (5)$$

$$\tilde{F}_2(\tilde{x},\tilde{y};P_{x1},P_{y1},s) = \frac{1}{2}\left[\frac{a'(s)}{a(s)} - C(s)\right]\tilde{x}^2 + \frac{1}{2}\left[\frac{b'(s)}{b(s)} + C(s)\right]\tilde{y}^2 + \sqrt{\varepsilon_T}\left[\frac{\tilde{x}P_{x1}}{a(s)} + \frac{\tilde{y}P_{y1}}{b(s)}\right]. \quad (6)$$

In Eqs. (5) and (6) prime denotes derivative with respect to $s$, $C(s) = \sin[2\theta(s)]\sqrt{\kappa_z(s)}\left(k_{0y}^2 - k_{0x}^2\right)/2k_0^2$, the constant $\varepsilon_T > 0$ is an effective emittance, and $a(s)$ and $b(s)$ are the periodic functions solving the envelope equations

$$a''(s) - \left[C'(s) - C^2(s) - \alpha_y\right]a(s) - \frac{2K}{a(s)+b(s)} = \frac{\varepsilon_T^2}{a^3(s)}, \quad (7)$$

$$b''(s) + \left[C'(s) - C^2(s) - \alpha_x\right]b(s) - \frac{2K}{a(s)+b(s)} = \frac{\varepsilon_T^2}{b^3(s)}. \quad (8)$$

The twisted angle solves the differential equation

$$\frac{d\theta(s)}{ds} = \frac{a^2(s)\alpha_y(s) - b^2(s)\alpha_x(s)}{a^2(s) - b^2(s)}, \quad (9)$$

where

$$\alpha_x(s) = -\sqrt{\kappa_z(s)}/k_0^2\left\{k_{0x}^2\sin^2[\theta(s)] + k_{0y}^2\cos^2[\theta(s)]\right\} \quad (10)$$

and

$$\alpha_y(s) = -\sqrt{\kappa_z(s)}/k_0^2\left\{k_{0x}^2\cos^2[\theta(s)] + k_{0y}^2\sin^2[\theta(s)]\right\} \quad (11)$$

are the rotational flow velocities as defined in Eqs. (10) and (11) in Ref. [5]. The envelope equations (7)-(9) can be written in a form similar to the generalized envelope equations in the cold-fluid equilibrium theory by substituting $C(s) = \sin[2\theta(s)]\sqrt{\kappa_z(s)}\left(k_{0y}^2 - k_{0x}^2\right)/2k_0^2$ and Eqs. (10)-(11) into Eqs. (7) and (8). They are identical to the generalized envelope equations of $a(s)$, $b(s)$ and $\theta(s)$ in the cold-fluid equilibrium theory, except that the thermal emittance terms that appear on the right hand side of Eqs. (7) and (8) are zero in the cold-fluid equilibrium theory.

It follows that the Hamiltonian in the canonical variables $(x_1,y_1,P_{x1},P_{y1})$ is then expressed as

$$H_{1\perp}(x_1,y_1,P_{x1},P_{y1},s) = \frac{\varepsilon_T}{2}\left\{\frac{P_{x1}^2}{a^2(s)} + \frac{P_{y1}^2}{b^2(s)} + \frac{x_1^2}{a^2(s)} + \frac{y_1^2}{b^2(s)}\right\} + \frac{d\varphi(s)}{ds}\left(y_1P_{x1} - x_1P_{y1}\right), \quad (12)$$

where we have introduced and demanded $d\varphi(s)/ds \equiv b(s)/a(s)\{\theta'(s) - \alpha_x\} = a(s)/b(s)\{\theta'(s) - \alpha_y\}$. It is readily shown that $\mathrm{E} = x_1^2 + y_1^2 + P_{x1}^2 + P_{y1}^2$ is an exact single-particle constant of the motion for the Hamiltonian in Eq. (12). In the reminder of this section, we consider the following trial choice of the Vlasov equilibrium distribution function

$$f_b(x_1,y_1,P_{x1},P_{y1},s) = \frac{N_b}{\pi^2\varepsilon_T}\delta\left(x_1^2 + y_1^2 + P_{x1}^2 + P_{y1}^2 - \varepsilon_T\right) \quad (13)$$

where $df_b/ds = 0$, $\varepsilon_T = \text{const} > 0$ is an effective emittance, and $\delta(x)$ is the Dirac $\delta$ function. Because the quantity $\mathrm{E} = x_1^2 + y_1^2 + P_{x1}^2 + P_{y1}^2$ is a constant of motion, the distribution function defined in Eq. (13) is indeed a Vlasov equilibrium, i.e., $\partial f_b/\partial s = 0$ .

## STATISTICAL PROPERTIES

The distribution described in Eq. (13) has the following statistical properties. First, the density profile is consistent with the assumed density in Eq. (2), i.e., $n_b(\tilde{x},\tilde{y},s)= \varepsilon_T\left[a(s)b(s)\right]^{-1}\iint f dP_{x1}dP_{y1} = N_b/\pi a(s)b(s)$ inside the beam boundary.

Second, in the normalized units, the average (macroscopic flow) transverse velocity of the beam equilibrium described by Eq. (13) is given in the twisted coordinates by $\tilde{\mathbf{V}}_\perp = \varepsilon_T\left[n_b a(s)b(s)\right]^{-1}\int \tilde{\mathbf{v}}_\perp f dP_{x1}dP_{y1}$ or

$$\tilde{\mathbf{V}}_\perp = \left(\tilde{x}a'/a - \alpha_x\tilde{y}\right)\mathbf{e}_{\tilde{x}} + \left(\tilde{y}b'/b + \alpha_y\tilde{x}\right)\mathbf{e}_{\tilde{y}} . \quad (14)$$

The flow velocity in Eq. (14) is identical to the flow velocity derived by the cold-fluid theory [5].

As a third statistical property, the beam equilibrium has the effective transverse temperature profile

$$T_\perp(\tilde{x},\tilde{y},s) = \varepsilon_T\left[n_b a(s)b(s)\right]^{-1}\int\left(\tilde{\mathbf{v}}_\perp - \tilde{\mathbf{V}}_\perp\right)^2 f dP_{x1}dP_{y1}$$

$$= \varepsilon_T^2\left(a^2+b^2\right)/2a^2b^2\left(1-\tilde{x}^2/a^2-\tilde{y}^2/b^2\right) . \quad (15)$$

As the fourth property, the 4 times the rms emittance of the beam in the twisted frame is $\varepsilon_{th} = 4(\beta_b c)^{-1} \times\sqrt{\left\langle\tilde{x}^2\right\rangle\left\langle\left(\tilde{v}_x - \tilde{V}_x\right)^2\right\rangle} = \varepsilon_T$ .

Finally, the Vlasov elliptic beam equilibrium has two limiting cases which are well know. It recovers the familiar rigid-rotor Vlasov equilibrium [3] by setting the major-axis equal to the minor-axis of the beam ellipse. It also recovers the familiar constant-radius, uniform-density rigid-rotor Valsov equilibrium [2] by taking the limit of a uniform magnetic field with $B_z = B =$ const.

## EXAMPLE

As an example, we consider a relativistic elliptic beam with $V_b = 198.5$ keV , current $I_b = 85.5$ A, aspect ratio $a/b = 5$ , and non-axisymmetric periodic permanent magnet focusing with $B_0 = 2.4$ kG , $S = 2.2$ cm , and $k_{0y}/k_{0x} = 1.52$ . [We propose to use it in a 10 MW L-Band ribbon-beam klystron (RBK) for the International Linear Collider (ILC).] For such a system the matched solution of the generalized envelope equations (7)-(11) is calculated numerically with the corresponding parameters: $k_{0x} = 1.57$ cm$^{-1}$ , $\sqrt{\kappa_{z0}} = 0.732$ cm$^{-1}$ , and $K = 1.13\times10^{-2}$ . As shown in Fig. 2, the solid lines represent the beam semi-axis envelopes with zero temperature which is corresponding to a cold beam, while the dashed lines represent the beam envelopes and twisted angles with 5 keV on-axis temperature.

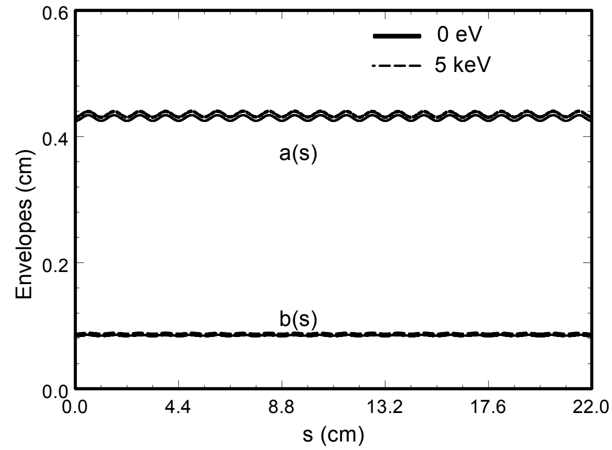


Figure 2: Plots of (a) envelopes $a(s)$ and $b(s)$ versus the axial distance $s$ for the relativistic twisted elliptic beam.

## SUMMARY

The single-particle Hamiltonian of a periodically twisted large-aspect-ratio elliptic beam in a non-axisymmetric periodic magnetic focusing field has been investigated. A new constant of motion has been found such that the self-consistent beam equilibrium can be constructed as a function of the constant of motion. The beam envelope equations and flow velocity equations have been derived. They are consistent with the generalized envelope equations derived from the cold-fluid equilibrium theory [5] when the temperature is taken to be zero. Statistical properties of the present Vlasov elliptic beam equilibrium have been studied. For current applications of interest, the temperature effects have been found to be small on periodically twisted large-aspect-ratio elliptic beams.

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