

COMPUTER SIMULATION OF EQUILIBRIUM ELECTRON BEAM DISTRIBUTION IN THE PROXIMITY OF 4TH ORDER SINGLE NONLINEAR RESONANCE

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Abstract

The beam distribution of particles in a storage ring can be distorted in the presence of nonlinear resonances. Computer simulation is used to study the equilibrium distribution of an electron beam in the presence of a single 4th order nonlinear resonance in a storage ring. Its result is compared with that obtained using an analytical approach by solving the Fokker-Planck equation to first order in the resonance strength. The effect of resonance on quantum lifetime of electron beam is also compared and investigated.

INTRODUCTION

As a particle moves in a storage ring, its orbit depends sensitively on its betatron tune. If the betatron tune is near a resonance condition, $n\nu_x + l\nu_y = m$, where m, n, l are integers, particles in the accelerator can encounter coherent kicks. Thus, its orbit will be perturbed and the beam distribution in phase space will be distorted from a simple ellipse. The degree of distortion depends on how close the resonance is and also the strength of resonance.

In 1-dimensional case, near the resonance $n\nu = m$, the Hamiltonian of the particle can be generally expressed as

$$KR = \left(\nu - \frac{m}{n}\right)J + D_\nu(J) + f_1(\phi, J) \quad (1)$$

where (ϕ, J) are the phase space angle and action variables, $D_\nu(J)$ is the detuning function, and $f_1(\phi, J)$ is some resonance structure function. In an electron storage ring, synchrotron radiation complicates the particle motion because it introduces radiation damping and quantum diffusion to the electron motion. To obtain the equilibrium beam distribution, these effects must be taken into account. In the absence of resonance, the equilibrium distribution is Gaussian in x and p_x . This distribution can be expressed with action-angle variables (ϕ, J) as exponential in J , i.e. $\psi(\phi, J) = e^{-J/J_0}$, where J_0 is the normal beam emittance determined by a balance between the radiation damping and quantum diffusion.

Recently, Fokker-Planck equation near a single resonance for an electron beam in a storage ring was solved to first order in resonance strength [1]. This gives an equilibrium beam distribution as

$$\Psi(\phi, J) = \exp \left[-\frac{1}{J_0} \left(J + \frac{f_1(\phi, J)}{\nu - \frac{m}{n} + D_\nu(J)} \right) \right] \quad (2)$$

When the perturbation is given by a magnetic multipole, $D_\nu(J)$ and $f_1(\phi, J)$ can be expressed as

$$D_\nu(J) = \frac{\varepsilon}{\pi} \left(\frac{J\beta}{2} \right)^{n/2} \frac{n!}{2 \cdot [(n/2)!]^2} \quad \text{if } n = \text{even}, \quad (3)$$

$$D_\nu(J) = 0 \quad \text{if } n = \text{odd}.$$

and

$$f_1(\phi, J) = \frac{\varepsilon}{\pi} \left(\frac{J\beta}{2} \right)^{n/2} \cos n\phi \quad (4)$$

$$\varepsilon = \frac{\ell}{n!(B\rho)} \frac{\partial^{n-1} B_y}{\partial x^{n-1}}$$

where $B\rho$ is the magnetic rigidity, and ℓ is the length of the multipole.

The quantum life time of the electron beam is also obtained in this calculation. It was shown that in the presence of a nearby nonlinear resonance, compared with the case away from resonance, the quantum lifetime of an electron beam is shortened by a factor of

$$\exp \left[\frac{|e|A^n}{\pi 2^n \left(\nu - \frac{m}{n} \right) \frac{\sigma_{x0}^2}{\beta} + \frac{1}{2} \frac{nm!}{[(n/2)!]^2} \mathcal{E} \sigma_{x0}^2 A^{n-2}} \right] \quad (5)$$

In this paper, a computer simulation is carried out in the proximity of 4th order nonlinear resonance to examine this theoretical model.

COMPUTER SIMULATION

In order to perform a computer simulation, the equation of motion including all of the relevant effects must be developed first. In our approach, the thin lens approximation is used, while collective effects due to the beam-generated electromagnetic fields are neglected. At any one point of the storage ring the beam transfer matrix for the transverse phase space coordinates between the n th turn and its previous turn are related by

$$\begin{bmatrix} x \\ x' \end{bmatrix}_n = M_{n,n-1} \begin{bmatrix} x \\ x' \end{bmatrix}_{n-1} \quad (6)$$

$$M_{n,n-1} = \begin{bmatrix} \cos 2\pi\nu + \alpha \sin 2\pi\nu & \beta \sin 2\pi\nu \\ \frac{1+\alpha^2}{\beta} \sin 2\pi\nu & \cos 2\pi\nu - \alpha \sin 2\pi\nu \end{bmatrix}$$

where ν is the betatron tune, β is the betatron amplitude, $\alpha = -\beta'/2$. In our simulation, the effects from the radiation damping and quantum diffusion caused by synchrotron radiation, are represented by a damping kick

$$\Delta x' = -\delta x', \text{ where } \delta = 2T_0/\tau$$

and a quantum excitation kick,

$$\Delta x' = q R,$$

where

$$q = \sqrt{6\delta \frac{\sigma_{x0}^2}{\beta_x^2}}$$

and R is a random number uniformly distributed between -1 and 1 ; σ_{x0} is the initial beam size in the x direction, T_o is the revolution period of the electron beam around the ring, and τ is the betatron radiation damping time. For studying the effects in the presence of a nonlinear resonance, a nonlinear kick $\Delta x' = kx^{n-1}$ is also added in the simulation, where k represents the kicking strength.

For the simulation in this study, we have assumed the perturbation is given by a 4th order magnet multipole error at a location $s=0$. As the electron circulates around the ring this multipole error gives δ -function kicks to the electrons once each turn. A random-generated beam with bi-Gaussian distribution in x and p_x and beam size of $\sigma = 400 \mu\text{m}$ is used as the initial beam distribution. The radiation damping time is set to be 10 ms. The electrons are simulated for tens of thousands revolution turns until they reach the equilibrium state, which is assumed when the rms of the beam size becomes statistically unchanged turn by turn. In this study 50,000 particles and typically more than 300,000 turns were used for various kicking strengths.

RESULT AND DISCUSSION

Equilibrium distribution near a 4th order nonlinear resonance

A comparison between the solved Fokker-Planck equation under the smooth approximation for an electron beam near a single 4th order nonlinear resonance and the simulation result is discussed here. In Fig. 1, the phase space distribution in p_x and x when $k=1000/\text{m}^3$ is shown. In the plot the lines showing the separatrix and the green points showing the stable and unstable fixed points as calculated from the Hamiltonian in eqn. (1) and $D_i(J)$ and $f_i(\phi, J)$ in eqns. (3) and (4). A good agreement can be seen. The same result is shown in the phase space plot in the action-angle variables, Fig. 2. In Fig. 3, an attempt is made to compare the distribution $\Psi(J)$ versus J between the calculation and the simulation result. The distribution of $\Psi(J)$ is obtained by integrating the particle counts over $\phi = -180^\circ$ to $\phi = 180^\circ$ for each J interval. It shows a large discrepancy at J where the nonlinear resonance islands should appear. This difference can not be explained simply by the constraints described in the reference (1). While the distribution for the islands in the simulation show a broad bump, the theoretical model gives a much thinner bump. Particularly, in the simulation the electron distribution in the resonance islands gives a long tail but the tail from the model drops quickly. For many different

kicking strength, they all exhibit the same trend. A study is also made to see if this discrepancy is due to the choice of our random number scheme by replacing the uniform (-1,1) scheme by a Gaussian scheme. The result shows that there is no obvious difference between these two different schemes. Thus, the discrepancy between the simulation and the first order theory is likely to be due to higher order effects of the nonlinear multipole strength, which will be pursued further later.

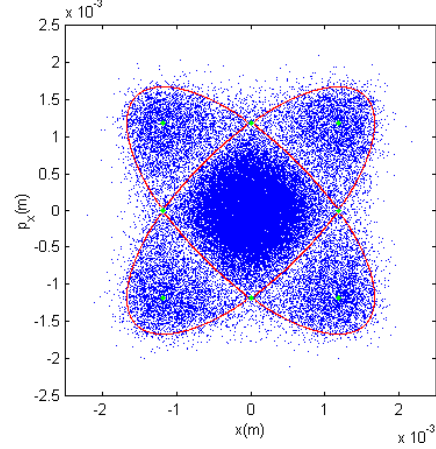


Fig. 1: The phase space distribution of the normalized coordinates (x, p_x) for $k = 1000/\text{m}^3$ near a 4th order nonlinear resonance. The points are from the simulation result when electron beam reaches equilibrium distribution. The four islands are seen clearly. The red curves show the separatrix and the green dots show the fixed points.

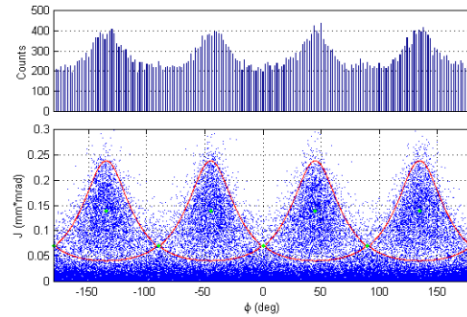


Fig. 2: The bottom plot shows the phase space distribution in action-angle variables (J, ϕ) . Conditions are the same as in Fig. 1. The top plot is the integrating of the counts along J for the same interval of ϕ , i.e. $\Psi(\phi)$.

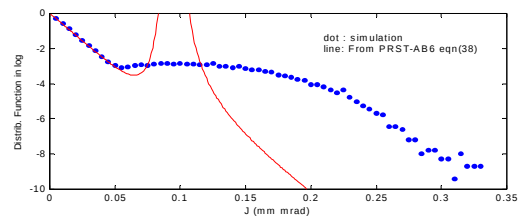


Fig. 3: Comparison of the beam equilibrium distribution vs. J , when $k = 1000/\text{m}^3$.

Near fractional tune=0.25 with negative k and far away from resonance

Due to the above discrepancy between the results from the model and that from the simulation, checks are performed for negative values of k , and also in regions far away from resonance. When k is set to be negative, the nonlinear islands will not be produced in the phase space when fractional tune ν_x is set to be above 0.25. In Fig. 4, both the results from the model and from the simulation are shown for $k = 0, -1000, -10000/\text{m}^3$. The results show good agreement, but then the nonlinear resonance effects are not pronounced in any case. In a detailed inspection of the result, one finds that there is a tail beyond $J > 0.15 \text{ mm}^2/\text{mrad}$ when k is large.

Fig. 5 shows the distribution $\Psi(J)$ with kicking strength $k = 5000/\text{m}^3$ and at different tunes far away from the 4th order nonlinear resonance. The agreement is also good. This can be attributed to a factor, $\nu\text{-}m/n$ in the denominator of eqn. (2).

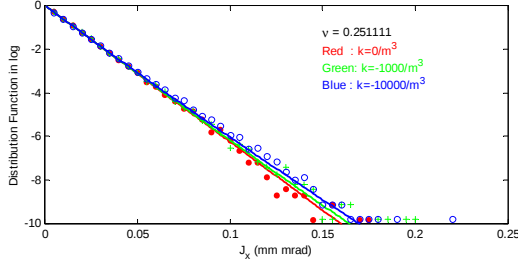


Fig. 4: Distribution $\Psi(J_x)$ versus J_x for several negative k 's near a 4th order nonlinear resonance.

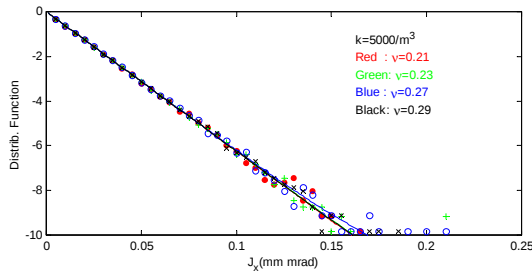


Fig 5: Distribution $\Psi(J_x)$ versus J_x for $k = 5000/\text{m}^3$ for different tunes far away from the 4th order nonlinear resonance.

Quantum lifetime

One of the important studies is to compare the quantum lifetime shortening factor calculated with eqn. (5) and that from the simulation near the 4th order nonlinear resonance. In Fig. 6 there are 3 sets of quantum lifetimes obtained from eqn. (5) and that from the simulation with $k = 0, -1000$ and $-5000/\text{m}^3$. It shows the quantum lifetime predicted by the model and that from the simulation at $J < 0.15 \text{ mm}^2/\text{mrad}$ has only about 10% difference. It agrees

within the model accuracy. At larger J , especially at larger k , the predicted quantum lifetimes are larger. This is related to the tail in the beam distribution in the simulation results when J is larger ($J > 0.15 \text{ mm}^2/\text{mrad}$) and k is larger ($k > 5000/\text{m}^3$). As for the positive k kicking strength, a continued study is underway. A comparison with other theoretical models, particularly in Ref. [2], will also be made.

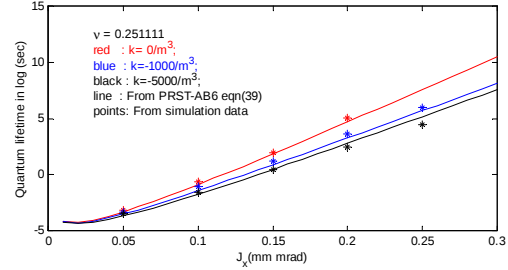


Fig. 6: The quantum lifetime vs J_x at $n=0.251111$ for kicking strength $k = 0, -1000, -5000/\text{m}^3$.

CONCLUSION

We have made a simulation study on the electron beam equilibrium distribution and effects on the quantum lifetime in the proximity of a 4th order nonlinear resonance. When the simulation results are compared with the theoretical calculation obtained by solving the Fokker-Planck equation to the first order, it shows that the model calculation is not able to predict correctly the electron beam equilibrium distribution, especially in the resonance islands region. While the model predicted quantum lifetime is reasonably good at lower kicking strength, it still has large discrepancy at larger J and larger kicking strength. A further study to improve the theoretical model is still needed and is under study.

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