

NONSINGULAR INTEGRAL EQUATION FOR STABILITY OF A BUNCHED BEAM *

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Abstract

The linearized Vlasov equation for longitudinal motion of a bunched beam leads to a singular integral equation, the singularity being associated with the tune spectrum of the single-particle motion. A discretization for numerical solution of the equation in this form is not well justified. A simple change of the unknown function gives an equation that can more readily be approximated by a matrix equation. In contrast to the usual approach (Oide-Yokoya) the equation for eigen-frequencies does not have a continuum of solutions corresponding to single-particle frequencies, but only a few solutions corresponding to coherent modes.

1 INTRODUCTION

The linearized Vlasov equation has long been considered the basic tool for determining the current threshold for microwave instabilities. Here we treat longitudinal motion only. A mode decomposition leads to an equivalent integral equation, which can be completely analyzed in the case of a coasting beam. For the bunched beam the modes are all coupled, and there are still aspects of the problem that are not very clear. Different forms of the integral equation have been discussed, for instance by Sacherer [1], Wang and Pellegrini [2], and Oide and Yokoya [3].

The authors of Ref. [3] made an important advance when they linearized the Vlasov equation about the proper equilibrium distribution determined by Haïssinski's equation. They then transformed to action-angle coordinates of the corresponding distorted potential well, and took Fourier transforms in time and angle. The resulting integral equation is replaced by a finite-dimensional matrix equation, by discretizing integrals on an action mesh, and truncating the azimuthal mode set. This scheme has been used to find the current thresholds of instabilities by several authors [4, 5, 6]. Some success in agreement of thresholds with tracking studies has been reported, but some difficulties have been noticed as well [4]. Convergence of the finite-dimensional approximation is in doubt, and the presence of incoherent mode frequencies, often degenerate with coherent modes, confuses the physical interpretation of eigenvectors.

Our intention here is to improve the calculational strategy by reformulating the Oide-Yokoya equation so as to eliminate the continuous spectrum and also guarantee con-

vergence under refinement of the action mesh and expansion of the azimuthal mode set. We approached this problem from the viewpoint of functional analysis, after noticing that the Oide-Yokoya integral equation is an integral equation of the third kind, which does not admit direct approximation by a finite-dimensional problem. The usual Fredholm equation of the second kind does admit a finite dimensional approximation, a property that stems from compactness of the Fredholm integral operator [7].

One approach to approximation of a third-kind problem is to transform it to a second-kind Fredholm problem. Such a transformation was made by Bart and one of the authors [8]. Recently we noticed a simpler approach, which is to transform the equation so that the new integral operator is compact, but not of standard Fredholm form. A compact operator on a Banach space is one that takes any bounded set into a relatively compact set. Roughly speaking, this implies that it has a smoothing action on a set of functions that are merely bounded, hence potentially noisy. We cannot give mathematical details here, but it is easy to see the value of our transformation in pedestrian terms: the discretized Oide-Yokoya operator has matrix elements that become unbounded as the mesh is refined, whereas all matrix elements from our transformed operator are bounded under mesh refinement.

2 OIDE-YOKOYA INTEGRAL EQUATION

The Vlasov equation for longitudinal motion in a linear RF bucket is

$$\frac{\partial f}{\partial \theta} + p \frac{\partial f}{\partial q} - (q + F(q, f)) \frac{\partial f}{\partial p} = 0. \quad (1)$$

This governs the distribution $f(q, p, \theta)$ depending on the normalized time and phase space variables

$$\theta = \omega_s t, \quad q = \frac{z}{\sigma_z}, \quad p = -\frac{E - E_0}{\sigma_E}, \quad \frac{\omega_s \sigma_z}{c} = \frac{\alpha \sigma_E}{E_0}, \quad (2)$$

where ω_s is the synchrotron frequency, z is the distance from the synchronous particle (positive in front), σ_z and σ_E are rms spreads of a low-current bunch, and α is the momentum compaction. The collective force is expressed in terms of the wake function W (positive for energy gain) and a normalized current parameter $I = e^2 N / (2\pi \nu_s \sigma_E)$ by

$$F(q, f) = I \int dq' W(q - q') \int dp f(q', p, \theta). \quad (3)$$

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The bunch population is N and ν_s is the synchrotron tune. When Fokker–Planck (FP) terms to account for synchrotron radiation are added to the right side of (1), we get an equation which has a unique equilibrium solution (under restrictions on the current and properties of the wake), which is also an equilibrium of the Vlasov equation. We linearize (1) about that equilibrium, but do not include the FP terms in the following analysis. The equilibrium has the form $f_0(q, p) = \exp(-p^2/2)\rho_0(q)/\sqrt{2\pi}$ where $\rho_0(q)$ is the solution of the Haïssinski equation. The Hamiltonian of the equilibrium motion, a functional of f_0 , is

$$H_0 = \frac{1}{2}(p^2 + q^2) + H_c(q, f_0), \quad (4)$$

where H_c is the collective part of the Hamiltonian,

$$H_c(q, f) = - \int_q^\infty F(q', f) dq'. \quad (5)$$

Following [3], we perform a canonical transformation to exact angle–action variables (ϕ, J) of the equilibrium motion. The transformation is written as $q = Q(\phi, J)$, $p = P(\phi, J)$. Defining the perturbation f_1 , we linearize (1) about f_0 as follows:

$$f(\phi, J, \theta) = f_0(J) + f_1(\phi, J, \theta), \quad (6)$$

$$\frac{\partial f_1}{\partial \theta} + \Omega(J) \frac{\partial f_1}{\partial \phi} - f'_0(J) \frac{\partial H_c(Q, f_1)}{\partial \phi} = 0, \quad (7)$$

$$\Omega(J) = H'_0(J), \quad (8)$$

$$f_0(J) = Ae^{-H_0(J)} \sim e^{-J}, \quad J \rightarrow \infty. \quad (9)$$

We next perform a Laplace transform of (7) with respect to θ , and a Fourier transform with respect to ϕ . The result is

$$(s + im\Omega(J))\hat{f}_1(m, J, s) - \check{f}_1(m, J, 0) - f'_0(J) \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{-im\phi} \frac{\partial}{\partial \phi} H_c(Q(\phi, J), f_1) = 0, \quad (10)$$

with \hat{f}_1 the double transform and \check{f}_1 the initial–value term,

$$\hat{f}_1(m, J, s) = \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{-im\phi} \int_0^\infty d\theta e^{-s\theta} f_1(\phi, J, \theta),$$

$$\check{f}_1(m, J, 0) = \frac{1}{2\pi} \int_0^{2\pi} e^{-im\phi} d\phi f_1(\phi, J, 0).$$

The integral on ϕ is zero for $m = 0$, from which it follows that the zero mode does not vary in time under the linearized Vlasov dynamics. Consequently, there is no reason to include a zero mode in studying growth of the perturbation. Henceforth $\hat{f}_1(0, J, s) = 0$ and $m \neq 0$ in (10).

The force F involves $\int dq' dp' \dots$, which we replace by $\int d\phi' dJ' \dots$. In view of the symmetry $Q(\phi, J) = Q(-\phi, J)$ and the definition $Q_1 = \partial Q / \partial \phi$, Eq.(10) takes the form (with $m, m' \neq 0$)

$$(\omega - m\Omega(J))\hat{f}_1(m, J, \omega) - i\check{f}_1(m, J, 0) + \sum_{m'} \int dJ' K(m, J, m', J') \hat{f}_1(m', J', \omega) = 0, \quad (11)$$

$$K(m, J, m', J') = -\frac{If'_0(J)}{2\pi} \int d\phi \sin m\phi \int d\phi' \cos m'\phi' \cdot Q_1(\phi, J) W(Q(\phi, J) - Q(\phi', J')). \quad (12)$$

For contact with Fourier analysis we have put $s = -i\omega$ and have written $\hat{f}_1(\cdot, \omega)$ for the previous $\hat{f}_1(\cdot, s)$. The Laplace transform is assumed to exist for $\text{Re } s \geq v_0$, hence for $\text{Im } \omega \geq v_0$. Eq.(11) is equivalent to the equation of Oide–Yokoya, when $\text{Im } \omega = 0$. Unstable modes correspond to poles of \hat{f}_1 at $\omega = u + iv$, $v > 0$.

If ω is real and in the range of $m\Omega(J)$, the equation is an integral equation of the third kind [8], owing to a zero of the factor $\omega - m\Omega(J)$. Such an equation may have solutions in a space of generalized functions, with delta–function and/or principal–value integral functionals concentrated at the zero. The delta – PV functionals correspond to the van Kampen modes [9]. In general the equation has no continuous solution. Thus, if we try a simple discretization of the J –integration to create a matrix equation, we shall be trying to represent a delta function numerically, which is hardly a worthy ambition. One could take v positive to get a regular integral equation, but at the small v needed to determine thresholds it would still have doubtful value for numerical work because “near zeros” of $\omega - m\Omega(J)$ would still be felt.

3 REGULARIZED EQUATION

The transformation to regularize the equation is remarkably simple. We merely redefine the unknown function to be g , with

$$g(m, J, \omega) = e^{J/2}(\omega - m\Omega(J))\hat{f}_1(m, J, \omega). \quad (13)$$

The exponential factor is inessential, being merely a convenience to symmetrize the asymptotic behavior of the kernel in J, J' . Now the integral equation for g at real ω is defined in terms of the limit

$$g(m, J, u) - ie^{J/2}\check{f}_1(m, J, 0) + \lim_{v \rightarrow 0^+} \sum_{m'} \int dJ' \frac{H(m, J, m', J')g(m', J', u)}{u + iv - m'\Omega(J')} = 0,$$

$$H(m, J, m', J') = e^{J/2}K(m, J, m', J')e^{-J'/2}. \quad (14)$$

If $H(m, J, m', J')g(m', J', u)$ has some minimal smoothness as a function of J' (for instance, satisfies a Hölder condition [10]), then the limit clearly exists if $\Omega(J)$ has locally linear behavior in a region near a point J_* where $u - m\Omega(J_*) = 0$. By Plemelj’s theorem [10] (the usual “PV $\pm i\pi\delta$ ” rule extended to functions that need not be analytic), the limit of the integral is then

$$P \int dJ' H(m, J, m', J') \frac{g(m', J', u)}{u - m'\Omega(J')} - \frac{\pi i H(m, J, m', J_*) g(m', J_*, u)}{|m'|\Omega'(J_*)}. \quad (15)$$

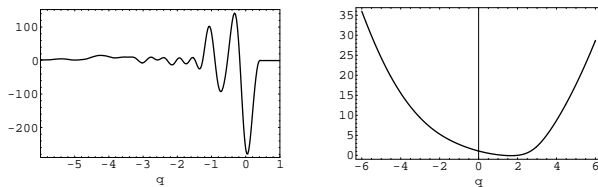


Figure 1: Wake function (V/pC) and distorted well

Now notice that (15) is smooth in J , and falls off nicely as m and J go to infinity, given reasonable smoothness of the wake function and minimal assumptions on g . This is a point that we emphasize strongly, since it is the clue to showing that the integral–sum operator of (14) is compact on a suitable function space. It has an “improving” effect on the functions to which it is applied, regarding their smoothness and asymptotic behavior.

4 NUMERICAL METHOD

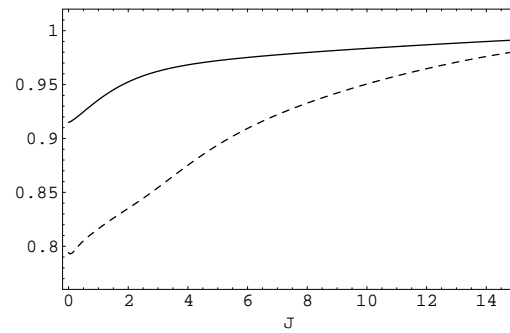
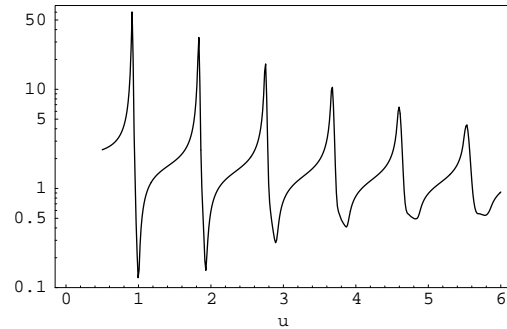
For numerical computation we discretize the equation for $g(m, J, \omega)$ on a mesh in J , and truncate the sum on m . We set up the discretization so that it is valid for $\omega = u + iv, v \geq 0$. This gives a matrix $\mathbf{1} + \mathbf{A}$ applied to a vector with components $g(m, J_i, \omega)$, the J_i forming a mesh corresponding to equally spaced values of $\sqrt{J_i}$. Poles in the upper half ω –plane, which may be arbitrarily close to the real axis, are then found by looking for zeros of the determinant of the linear system:

$$\det(\mathbf{1} + \mathbf{A}(\omega, I)) = 0. \quad (16)$$

In contrast to the Oide–Yokoya method, we do not get a linear eigenvalue problem, but rather the nonlinear equation (16) for finding coherent modes. Starting at low current I there will be no zero in the upper half–plane, and at a critical current a zero for the most unstable mode will cross the real axis.

To discretize the integral in (14) we first change to $y = \sqrt{J}$ as the variable of integration, getting a Jacobian that vanishes at $y = 0$. At ω/m' such that the denominator is never small, we apply Simpson’s rule. In the contrary case we represent the numerator and denominator locally as quadratic functions, obtained by interpolation of mesh values, then carry out the local integral analytically. We take care to keep the minimum of the denominator away from the endpoints of the local integral, except for the integration near $y = 0$ in which case the small Jacobian cancels the small divisor.

As an example we take Bane’s wakefield for the SLC damping ring, and the corresponding distorted well, as shown in Fig.(1). The single–particle frequency $\Omega(J)$ is shown for two currents in Fig.(2). The absolute value of the determinant is plotted on a log scale in Fig.(3) as a function of u at $v = 0.01$, for a current of $I = 0.03$ pC/V. The downward fingers, representing incipient unstable coherent modes, ought to reach zero at various values of I


 Figure 2: Ω at $I=.03$ (upper), $I=.0828$ (lower) pC/V

 Figure 3: $|\det(\mathbf{1} + \mathbf{A}(u + iv, I))|$ vs. u

when we increase I . It looks as though the frequency of the quadrupole mode will be about $1.82\omega_s$, in agreement with the value found in simulations and measurements. The code is too new to allow confidence in the quantitative accuracy of Fig.(3), but we anticipate that the qualitative picture will be similar after the code is refined and validated.

5 REFERENCES

- [1] F. Sacherer, IEEE Trans. Nucl. Sci. **NS-24**, 1393 (1977).
- [2] J.M. Wang and C. Pellegrini, *Proc. XI Intl. Conf. High Energy Accel.*, Geneva, 1980, p.554.
- [3] K. Oide and K. Yokoya, KEK Preprint 90-10 (1990); K. Oide, KEK 90-168 (1990), KEK 94-138 (1994).
- [4] M. D’yachkov, Ph.D. Thesis, U. of British Columbia (1995).
- [5] A. Mosnier, Nucl. Instr. Meth. Phys. Res. A **438**, 225 (1999).
- [6] K. Bane and K. Oide, *Proc. 1995 IEEE Part. Accel. Conf., Dallas, Texas; Proc. 1993 IEEE Part. Accel. Conf., Washington, DC.*
- [7] P. M. Anselone, *Collectively Compact Operator Approximation Theory and Applications to Integral Equations*, (Prentice–Hall, 1971).
- [8] G. R. Bart and R. L. Warnock, SIAM J. Anal **4**, 609 (1973); G. R. Bart, J. Math. Anal. and Appl. **79**, 48 (1981).
- [9] N. G. van Kampen, Physica **21**, 949 (1955).
- [10] N. I. Muskhelishvili, *Singular Integral Equations*, (Noordhoff, Groningen, 1958; Dover reprint, 1992).