SYMPLECTIC INTEGRATOR FOR PARTICLE TRACKING IN COMPLEX MAGNETIC FIELD

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Abstract

High field wigglers for synchrotron radiation production have usually rather complex distribution of magnetic field, and particle motion through it can hardly be treated analytically. This paper concerns a simple and reliable receipt for a symplectic algorithm to track a particle through such fields. Input data for numeric integration are taken directly from results of magnetic mapping or simulation of a 2D field values array. A 3-pole 7 T superconducting wiggler is considered as an example.

1 INTRODUCTION

The present work is devoted to practical realization of a simple and fast symplectic algorithm for numerical study of charged particle motion in a rather complex (3-Dimensional) magnetic field in a circular accelerator. As an example, we consider a high-field superconductive wiggler, installed in a synchrotron radiation source.

A coordinate transformation $F(\overline{q},\overline{p})$ of a dynamic system in a *2N*-dimension phase space is symplectic if [1]

$$M^T JM = J\,, \tag{1.1}$$

where *M* is the transformation Jacobian,

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{1.2}$$

and *I* is an *N*-dimensional unit matrix. As a consequence, det M=1.

Evolution of a Hamiltonian system in phase space is described by a symplectic coordinate transform and implies conservation of the Poincare integral invariants, which provides automatic validity of the Liouville theorem of conservation of the system phase volume.

Non-symplectic algorithms, for instance the well known Runge-Kutta method, do not provide det $M = 1$ and lead, at numerical integration of conservative systems, to artificial oscillation damping, which causes suppression of actual dynamical effects and appearance of non-existent ones.

2 INTEGRATION ALGORITHM

A sufficient number of canonical numerical integrators are available [2]. For our case we have chosen the simple and reliable Verlet scheme [3]. Let us consider the Hamiltonian of the system as

$$H(\overline{q},\overline{p}) = \sum_{i=1}^{N} \frac{p_i^2}{2} + V(\overline{q})\,. \tag{2.1}$$

According to Verlet, the change of the coordinates (q_i, p_i) at the instant t to the coordinates $(\overline{q}_i, \overline{p}_i)$ at the instant $t + \Delta t$ is made with the help of a canonical transformation

$$p_i = \frac{\partial F}{\partial q_i} = \overline{p}_i + \Delta t \frac{\partial H(q,\overline{p})}{\partial q_i}, \tag{2.2}$$
$$\overline{q}_i = \frac{\partial F}{\partial \overline{p}_i} = q_i + \Delta t \frac{\partial H(q,\overline{p})}{\partial \overline{p}_i}.$$

The generating function for such transformation is equal to $F(q,\overline{p},\Delta t) = q \cdot \overline{p} + \Delta t H(q,\overline{p})$. Earlier such a scheme was applied in consideration of particle motion in a helical undulator, whose magnetic field can be described analytically [3]. Below we take up realization of this algorithm for the case of an arbitrary magnetic field presented by a two-dimension array of values obtained via measurement or simulation.

Conventionally in the 6-dimension phase space $(x, p_x, z, p_z, -l, p_{-l})$, where x and z are the transverse coordinates, l is the path length and $p_{-l} = \Delta p / p_0$ is its conjugate momentum, the Hamiltonian of a relativistic particle has the form [4]:

$$H = -\frac{e}{p_0} A_s - \sqrt{(1 + p_{-l})^2 - \left(p_x - \frac{e}{p_0} A_x\right)^2 - \left(p_z - \frac{e}{p_0} A_z\right)^2}\,, \tag{2.3}$$

where $A_{x,z,s}$ are the components of the vector potential of magnetic field.

The main features of the proposed integration scheme are as follows:

- Hamiltonian (2.3) is written in simple terms of Cartesian reference frame on the contrary to the Serret-Frenet reference frame conventional for circular accelerators (see for instance [5]).
- The only simplification is expansion of the square root in (2.3) to the main order in x' and z'.
- Choice of the vector potential gauge $A_x = 0$ (this choice is convenient for our initial data, see below),
- The designations $u = eA_z / p_0$, $w = eA_s / p_0$ are introduced.

At the above-mentioned conditions, application of general scheme (2.2) to Hamiltonian (2.3) allows explicit conversion of the coordinate transformation for the first order of the step Δs:

$$\overline{z} = z + \Delta z = z + \Delta s\left(\frac{p_z - u + \Delta s \cdot w_z}{1 - \Delta s \cdot u_z}\right),$$
$$\overline{p}_z = p_z + \Delta z \cdot u_z + \Delta s \cdot w_z\,, \tag{2.4}$$

$$\bar{p}_x = p_x + \Delta z \cdot u_x + \Delta s \cdot w_x ,$$
$$\bar{x} = x + \Delta s \cdot \bar{p}_x ,$$

where $u_y = \partial u / \partial y$ and the same for *w*. For the sake of simplicity, expressions (2.4) are written for $p_{-l} = \Delta p / p_0 = 0$, however, it makes no problem to derive them for the general case of an arbitrary particle pulse, too.

One can see that the order of error of the algorithm presented is Δs^2. It is possible to continue construction of the integration algorithm for higher orders of Δ*s*, but in this case the complicated equations provide significant increase of the computation processor time.

3 INPUT DATA

Magnetic field of a superconductive wiggler has an intrinsic 3D distribution, which can hardly be described analytically and which is usually obtained either with the help of numerical simulation or via measurement. It seems natural and convenient to take results of measurement or simulation as an initial array of data for integration of the particle motion equations. Typically, distribution of wiggler magnetic filed is measured in a horizontal plane by array of *N* Hall probes, which is moved along the wiggler axis with some step. The obtained sequence of magnetic field values $B_z(x_i, z=0, s_j)$, $i = 1...N$, $j = 1...M$, where M is the number of steps, can be used for computation of the vector potential required to realize algorithm (2.4).

We will write the vector potential components for the case of a magnetic field with a horizontal plane of symmetry as follows [6]:

$$A_x = 0 \quad \text{(from the gauge condition)},$$
$$A_z(x,z,s) = \sum_{n,m=0}^{\infty} a_{z,n+1,2m+1}(s) \cdot x^{n+1} z^{2m+1} , \qquad (3.1)$$
$$A_s(x,z,s) = \sum_{n,m=0}^{\infty} a_{s,n+1,2m}(s) \cdot x^{n+1} z^{2m} .$$

The magnetic field can also be represented as the series:

$$B_z(x,z,s) = \sum_{n,m=0}^{\infty} b_{z,n,2m}(s) \cdot x^n z^{2m} , \qquad (3.2a)$$
$$B_s(x,z,s) = \sum_{n,m=0}^{\infty} b_{s,n,2m+1}(s) \cdot x^n z^{2m+1} , \qquad (3.2b)$$
$$b'_{z,n,2m} = (2m+1) b_{s,n,2m+1} ,$$

where *b' = db / ds*.

On the one hand, terms of series (3.2a) and their longitudinal derivatives, determining terms of series (3.2b), can be found numerically from the array of field magnitudes $B_z(x_i, z=0, s_j)$. On the other hand, the Maxwell equation $\vec{B} = rot\vec{A}$ allows one to link terms of (3.2a) and (3.2b) via recurrent relations:

$$(2m+1)(n+1) a_{z,n+1,2m+1} = b'_{z,n,2m} ,$$
$$(n+1) a_{s,n+1,2m} = -b_{z,n,2m} \qquad (3.3)$$

and to solve so the task of determination of the vector potential components via a discrete set of vertical magnetic field values measured (or computed numerically) in the median plane.

4 ALGORITHM APPLICATION

As an example we consider particle tracking in the field of a 3-pole superconductive wiggler with a maximal field of 7 T. Such a wiggler was designed and manufactured at BINP for an SR source [8]. The wiggler has 3 magnets and generates hard X-rays from the central one with the maximal field.

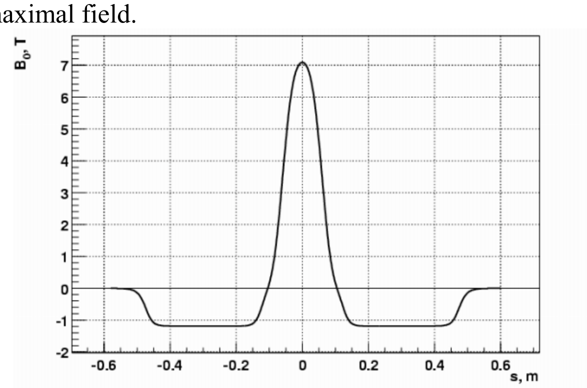

Fig.4.1 Vertical magnetic field along the wiggler axis.

Magnetic field of the wiggler was mapped with the help of an array of horizontally placed Hall probes, being moved along the wiggler axis in the median plane. The data array $B_z(x_i, z=0, s_j)$ was used as input values, as described above. The central Hall probe measures the dipole field distribution as seen from Fig.4.1.

The symplectic integrator was realized in the C++ language and was included in a code simulating particle motion in a circular accelerator. As a sample lattice we used a synchrotron radiation source at the energy of 1.7 GeV. The lattice consists of 8 identical DBA cells. The wiggler is placed into one of the dispersion-free straight section and, thus, breaks the 8-fold ring symmetry.

The array of actual measurements of the wiggler magnetic field (510 steps in the longitudinal direction) is used to construct a discrete set of the vector potential components. Beside the wiggler non-linearity, we take into account sextupole magnets to compensate natural chromaticity of the ring. Since the length of such magnets is much less than the wavelength of betatron oscillations, they can be tracked in a kick approximation. The horizontal phase–plane portrait of the particle initially perturbed by the chromatic sextupoles is shown in Fig.4.2. The particle was launched at some azimuth with a horizontal displacement and its motion was tracked during 10000 revolutions. The bulk of phase trajectories are regular ones and only in the edge of the dynamical aperture, there is the 8-order resonance and a weak stochastic layer around it. The horizontal dynamical aperture size is from –40 mm to +50 mm.

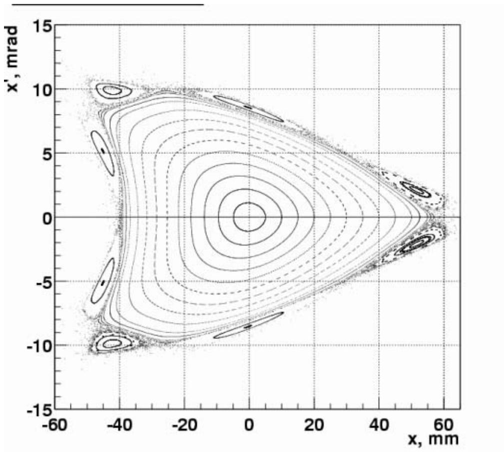

Fig.4.2 Horizontal phase trajectories of non-perturbed motion.

Fig.4.3 and Fig 4.4 show the structure of the horizontal phase space for the same conditions as in Fig.4.2 but with the superconducting wiggler switched-on. Phase trajectories in Fig.4.3 were obtained with the 4-order Runge-Kutta algorithm while Fig.4.4 shows the same plot but simulated with the symplectic algorithm described above. Both methods were applied to the magnetic field array with the same step.

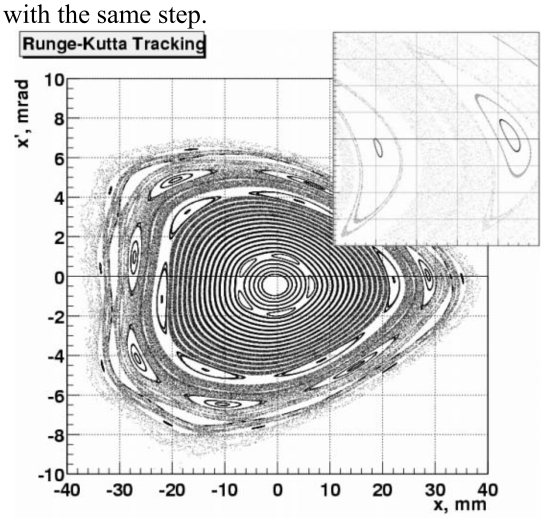

Fig.4.3 Runge-Kutta method applied to the SC wiggler perturbation.

Comparing the phase portraits, one can make a conclusion that the wiggler actually introduces rather strong perturbation into the particle motion: the dynamical aperture has reduced down to the size of ± 35 mm and there have appeared rather strong resonances of the 5-, 6- and 7-order inside it. Also one see that regular curves in Fig 4.4 are transformed to diffuse layers in Fig.4.3. This artificial effect looks like a rather strong stochastic component in the system under consideration.

Another important feature of the symplectic algorithm is the computation speed. While computation of a phase space of 10000 revolutions by the Runge-Kutta method requires 59 sec of the processor time (IBM PC Pentium 630MHz), application of the symplectic algorithm makes it possible to reduce this time down to 22 sec. For comparison, calculation of 10000 revolutions of a particle with the chromatic sextupoles needs only 1 sec of the processor time.

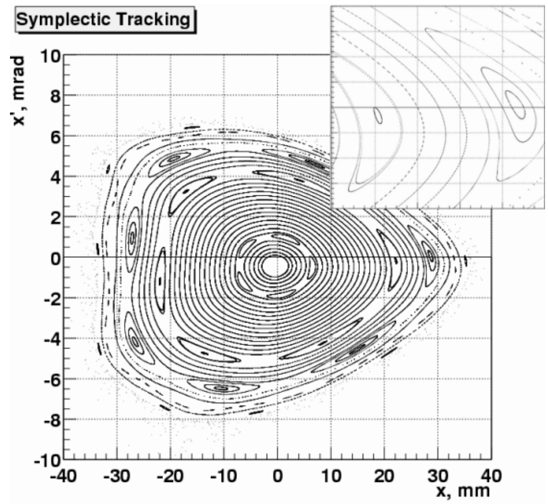

Fig.4.4 The same as Fig.4.3 but with the symplectic algorithm.

5 CONCLUSIONS

The symplectic integration method has been developed and realized, which allows study of motion of a relativistic particle in complex magnetic field of a super-conductive wiggler. Among the features of the method the following can be noted:

- The method conserves the system phase volume.
- The method works with the natural Cartesian wiggler reference frame.
- Expressions for the canonical transformations were derived explicitly (at the first order) and no numerical solution of implicit equations is required.
- Transformation equations are rather simple and provide high-speed computer realization.

6 REFERENCES

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