

A METHOD FOR NUMERICAL SOLUTION 2-D POISSON'S EQUATION WITH IMAGE FIELDS

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Abstract

Simulation of high intensity accelerators leads to the calculation of space charge forces between macroparticles in the presence of acceleration chamber walls. To calculate space-charge forces, one solves the Poisson's equation in 3-D. For an accelerator with long bunches, the 3-D problem is usually reduced to 2-D, because the transverse motion can be decoupled from the longitudinal due to very different characteristic times. The simulation of large macroparticle ensembles requires rapid evaluations of space-charge forces. Fast field solvers are usually based on Fourier-transform methods (FFT), but they are not applicable for arbitrary geometries of walls. For arbitrary walls, the solution for Poisson's equation can be derived using a Green function, which is numerically calculated on discrete mesh points. For simple wall geometries, e.g., circle and ellipse, it can be expressed analytically. Usually, such technique results in slow solvers. We discuss here a method, which combines a Green function technique and a FFT solver. In the first step, the Green function technique is used for arbitrary walls to calculate potentials on an intermediate rectangular boundary, which includes all macroparticles. In the second step, a FFT solver calculates potentials inside this rectangular domain with given potentials on its boundary. Preliminary numerical results will be reported.

1 INTRODUCTION

Simulation of high intensity accelerators leads to the calculation of space charge forces between macroparticles in the presence of acceleration chamber walls [1]. For an accelerator with long bunches, the 3-D problem is usually reduced to 2-D [2]. The simulation of large macroparticle ensembles requires rapid evaluations of space-charge forces. Fast field solvers are usually based on FFT methods, but they are not applicable for arbitrary geometries of walls. For arbitrary walls, the solution for Poisson's equation can be derived using a Green function, which is calculated on discrete mesh points. Usually, such technique results in slow solvers. We discuss here a method, which combines a precision of Green function technique and a high speed of FFT solvers. This method for a free space had been outlined in Ref [3] and its computation details had been discussed and illustrated in Ref. [4]. In this paper, this method is applied for arbitrary walls.

2 MATHEMATICAL ALGORITHM

2.1 From 3D to 2D Poisson problem

To calculate space-charge forces, one solves the Poisson's equation in 3D with boundary (wall) conditions:

$$\Delta U(x, y, z) = -\rho(x, y, z)/\epsilon_0. \quad (1)$$

An explanation to reduce 3D problem to 2D had been described in Ref. [2], considering an accelerator with long bunches, and assuming that the transverse motion is decoupled from the longitudinal. Then, the beam can be sliced in many longitudinal slices, and the space charge density, $\rho(x, y, z)$ can be approximately treated as [2]:

$$\rho(x, y, z) = \rho_{\parallel}(z) \cdot \rho_{\perp}(x, y) \quad (2)$$

The potential is also decomposed in a longitudinal and transverse part as $U(x, y, z) = U_{\parallel}(z) \cdot U_{\perp}(x, y)$. The problem is reduced to 2-D Poisson's equation:

$$\Delta U_{\perp}(x, y) = -\rho_{\perp}(x, y)/\epsilon_0. \quad (3)$$

2.2 Green function

The solution of Poisson equation for the fields due to space charge density $\rho_{\perp}(x_0, y_0)$ is given by [3,5,6]:

$$U_{\rho}(x, y) = \int_F \rho_{\perp}(x_0, y_0) G_{\perp}(x, y, x_0, y_0) dx_0 dy_0, \quad [4]$$

where $G_{\perp}(x, y, x_0, y_0)$ is the Green function, which satisfies to homogeneous Dirichlet conditions on the walls. The integration is performed on an area occupied by the space charge $dF = dx_0 dy_0$ (see Fig. 1).

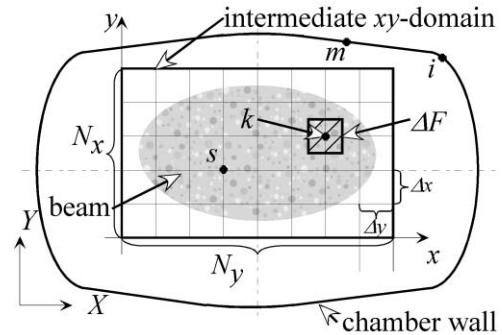


Figure 1: Arbitrary walls and the calculation domain.

Numerical calculations can be performed on a rectangular $N_x \times N_y$ -mesh. The Green function has to be calculated for every pair of points s and k and consists of N^2 elements ($N = N_x \times N_y$). The integral in Eq. (4) is approximated by the sums on the mesh. The potential at observation point, s is given by:

$$U_{\rho}^s = \sum_k \rho_k G_k \Delta F_k, \quad (5)$$

where $\Delta F = \Delta x_0 \Delta y_0$. A number of calculations is proportional to N^2 . At $N \geq 10^4$, the direct calculations of potential using Green function require a large computation time and memory to store Green function. Although Green function can provide precise solution for arbitrary walls, such technique results in slow solvers.

2.3 PIC method with Green function

We discuss here a method, which combines a Green function technique and a FFT solver. Let's define an intermediate rectangular xy -domain (see Fig. 1), which includes all macroparticles. At a preliminary step before beam dynamics simulations, one should prepare the Green function for calculations of potentials on the boundary of this xy -domain for a given wall geometry. The number of Green function elements being calculated and stored is proportional to $N \times N_b$, where N_b is the number of nodes on the boundary of xy -domain. N_b can be a quit small, $N_b \geq 8$, and number of Green function elements, $N \times N_b$ can be reasonable.

During beam dynamics simulations, the "particle-in-cell" (PIC) algorithm [7] can be used for calculations of beam self-fields. The space-charge density on the grid, ρ_k is calculated by allocating the contribution of each macroparticle to the grid of xy -domain. At an additional calculation step, the values of potential on the boundary of xy -domain are calculated according to Eq. (5) using stored Green function and current values of ρ_k . Then, a FFT solver calculates potentials inside this rectangular xy -domain at given potentials on its boundary. A further PIC procedure is not changed.

3 CALCULATIONS FOR DIFFERENT SHAPES OF BOUNDARIES

To test the presented method, we used MATHCAD [8] and its relaxation solver (RELAX) for a square domain. Note, that a standard FORTRAN FFT solver can be easily applied instead of it on such square or rectangular domain. For example, an appropriate solver, FPS2H using FFT techniques is available in "IMSL Math Library" by Visual Numerics [9].

Below the calculation results for different shapes of boundaries are presented. Calculations have been done for the round beam with uniform space-charge density $\rho_{\perp}(x_0, y_0)$. The potential distributions calculated directly by the summation (5) with an appropriate Green function on a total XY -area and on the boundary of the square xy -domain are denoted as U_{ρ}^{direct} and U_{ρ}^{square} , respectively. The potential distribution inside the square domain calculated with the RELAX solver is denoted as U_{ρ}^{mesh} . The values of relative deviation between the direct solution, U_{ρ}^{direct} and solution on square domain, U_{ρ}^{mesh} are denoted as ε_U .

3.1 Calculations in Free Space

The Green function for unlimited two dimensions (or Green function for free-space in 2D) is given by [3-6]:

$$G_{\text{FS}}(x, y, x_0, y_0) = -(2\pi\varepsilon_0)^{-1} \ln R, \quad (6)$$

$$\text{with } R = \sqrt{(x-x_0)^2 + (y-y_0)^2}. \quad (7)$$

The calculated potential distributions U_{ρ}^{direct} , U_{ρ}^{square} , and U_{ρ}^{mesh} at $\varepsilon_U \leq 0.3\%$ are shown in Fig. 2.

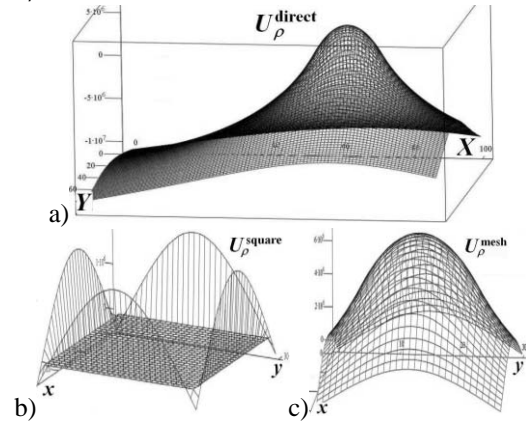


Figure 2: Potential distributions, U_{ρ}^{direct} (a), U_{ρ}^{square} (b), and U_{ρ}^{mesh} (c) for free-space.

3.2 Circle Shape

The Green function on the circle of radius a_0 is [10]:

$$G_{\text{C}}(x, y, x_0, y_0) = (2\pi\varepsilon_0)^{-1} \ln(\bar{R}r_0/Ra_0), \quad (8)$$

with $\bar{R} = \sqrt{(r_0^2x - a_0^2x_0)^2 + (r_0^2y - a_0^2y_0)^2} / r_0^2$, $r_0 = \sqrt{x_0^2 + y_0^2}$ and $R = \sqrt{(x-x_0)^2 + (y-y_0)^2}$.

The calculated potential distributions U_{ρ}^{direct} , U_{ρ}^{square} , and U_{ρ}^{mesh} at $\varepsilon_U \leq 0.15\%$ are shown in Fig. 3.

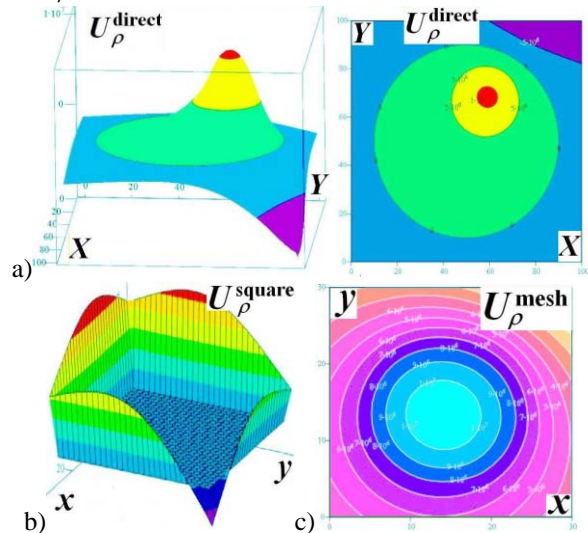


Figure 3: Potential distributions U_{ρ}^{direct} (a), U_{ρ}^{square} (b), and U_{ρ}^{mesh} for the circle walls.

3.3 Elliptic Shape

For the vacuum chamber of an elliptic shape with half width, w and half height, h , the Green function [5,11]

can be expressed in the elliptic coordinates (μ, ϑ) , which related to rectangular coordinates by the relations:

$$\begin{cases} x = F \cosh(\mu) \cos(\vartheta) \\ y = F \sinh(\mu) \sin(\vartheta) \end{cases}, \quad (9)$$

where F is the focus of the ellipses in rectangular system. The vacuum chamber can be described by the ellipse of coordinate $\mu = \mu_1$, with $w = F \cosh \mu_1$, $h = F \sinh \mu_1$. The Green function is given by [5,11]:

$$G_{\text{EL}}(\mu, \vartheta, \mu_0, \vartheta_0) = (4\pi\epsilon_0)^{-1} \{-2 \ln R + 2[\mu_1 + \ln(a/4)] - \Sigma\}, \quad (10)$$

where $\Sigma = \sum_{n=1}^{\infty} (4e^{-n\mu_1}/n) \times$
 $\times [\cosh(n\mu_0) \cos(n\vartheta_0) \cosh(n\mu) \cos(n\vartheta) / \cosh(n\mu_1) +$
 $+ \sinh(n\mu_0) \sin(n\vartheta_0) \sinh(n\mu) \sin(n\vartheta) / \sinh(n\mu_1)]$,
 and $a = 2w/\cosh \mu_1$ is the strip radius.

The calculated potential distributions U_{ρ}^{direct} , U_{ρ}^{square} , and U_{ρ}^{mesh} are shown in Fig. 4. The values of ϵ_U are less than 0.54%.

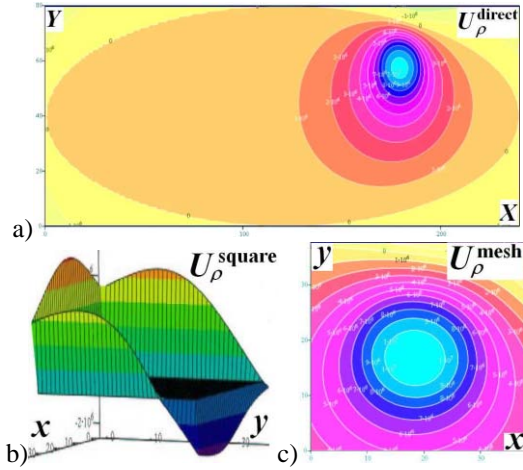


Figure 4: Potential distributions U_{ρ}^{direct} (a), U_{ρ}^{square} (b), and U_{ρ}^{mesh} (c) for elliptic walls.

3.4 Arbitrary Shape of Walls

For an arbitrary shape of walls, the “discrete” Green function, $\|G^{\text{D}}\|$ defined on the mesh can be used [6]. The calculation method is based on the integral equation method using image charges. $\|G^{\text{D}}\|$ is a square matrix with elements G_{sk}^{D} . The elements of the matrix $\|G^{\text{D}}\|$ relate the potential at the mesh node, s (the observation point) with a unit charge at the mesh node, k (the source point):

$$G_{sk}^{\text{D}} = \varphi_{sk} - \|\varphi_{si}\| \cdot \|\varphi_{mi}\|^{-1} \cdot \|\varphi_{mk}\|, \quad (10)$$

where $\|\varphi_{mi}\|$ is a square matrix, $\|\varphi_{si}\|$ is a row vector, $\|\varphi_{mk}\|$ is a column vector, while all elements of matrices, $\|\varphi_{\xi\eta}\|$ are the Green functions for a free space (6) calculated between corresponding source and observation points. For image charges one defines an additional mesh on the walls (see Fig. 1) with the observation points, m

and the source points, i . Singularities in (6) are avoided by a charge smoothing procedure [6].

Let’s take the vacuum chamber with racetrack shape of cross section as an example of an arbitrary shape of the walls. For the racetrack shape of walls, the potential distributions have been calculated using the “discrete” Green function $\|G^{\text{D}}\|$. The calculated potential distributions U_{ρ}^{direct} , U_{ρ}^{square} , and U_{ρ}^{mesh} are shown in Fig. 6. The values of relative deviations ϵ_U are less than 0.01%.

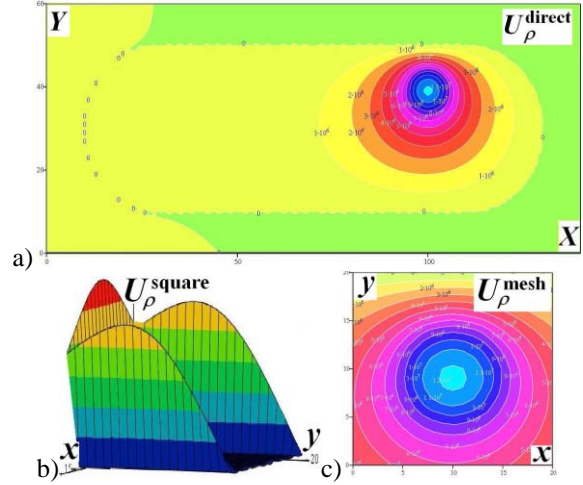


Figure 5: Potential distributions U_{ρ}^{direct} (a), U_{ρ}^{square} (b), and U_{ρ}^{mesh} for racetrack shape of walls.

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