

# INDUCTIVE IMPEDANCE AND LONGITUDINAL INSTABILITY \*

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## Abstract

Time domain study of the longitudinal bunch instability in a storage ring is performed for the very high frequency wake fields. These fields are usually described by the inductive impedance. The Fokker-Planck equation for the particle phase space distribution is solved by using the original implicit finite-difference method. A new developed quasi Green function is used for the wake potential reconstruction. It was found that the high frequency wake fields are responsible for the bunch instability, that has mainly "saw-tooth" character and is accompanied by bursts of coherent radiation, stochastically distributed in time.

## 1 INDUCTIVE IMPEDANCE AND WAKE FUNCTION

Inductive impedance  $Z^{\parallel}$  and inductive wake function  $w_{ind}$

$$Z^{\parallel}(\omega) = -iL\omega \quad w_{ind}(s) = -Lc^2\delta'(s)$$

are used for the description of the wake fields, responsible for the bunch self-acceleration, when the head of the bunch loses energy and the tail gets it back. In this case the bunch energy loss is considerably smaller than the energy spread in the bunch. The inductive wake potential  $W_{ind}(s)$  is just the derivative of the charge density distribution  $\rho(s)$

$$W_{ind}(s) = \int_{-\infty}^s w_{ind}(s-s')\rho(s')ds' = -Lc^2\frac{\partial}{\partial s}\rho(s) \quad (1)$$

Loss factor of the inductive wake potential is zero and energy spread can be easily calculated for the bunch of the Gaussian shape with the longitudinal size  $\sigma_z$  and total charge  $q$

$$\Delta E_{ind} = q\frac{Lc^2}{\sigma_z^2}\left(\frac{1}{6\pi\sqrt{3}}\right)^{1/2}$$

Inductive impedance is the main part of impedance of the vacuum chamber for different storage rings [1]. Bellows, tapers, pumping slots, masks, BPMs are considered to be the inductive elements. The inductive approach is used for the description of the fields in the resistive or rough wall vacuum chambers. How far can the inductive description of the accelerating elements in the storage rings be extended?

## 2 WAKE POTENTIALS

Wake field calculations for short bunches can give the answer for these questions. We present two examples: a bellow (Fig. 2) and shielded bellow (Fig. 1) Some other examples of the wake potentials of different accelerator elements can be found in Ref. [2],[3]. Wake field study shows that

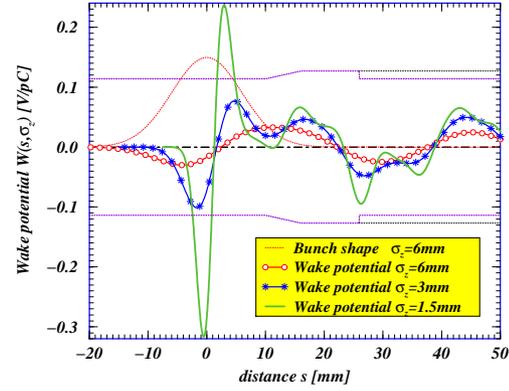


Figure 1: Wake potentials for a shielded bellow.

the inductive character of the wake potential can be transformed to the resistive one, in the long non-homogenous tubes [3]. Inductive description for the wake potentials is limited by the short range, high frequency fields. Dirac function, cavity short range function or resonator wake function can be the wake functions for the inductive wake potentials. Resonator wake function  $w_r$  is

$$w_r(s) = -W_r\frac{1}{k_Q}\frac{d}{ds}[\exp(-\frac{k_r}{2Q}s)\sin k_Qs]$$

where  $k_Q = k_r\sqrt{1-1/4Q^2}$ . This simple function gives good enough description for the inductive wake potentials of the long bunches ( $k_r\sigma_z \gg 1, Q \gg 1$ )

$$W(s) \approx -W_r\int_{-\infty}^s \cos k_r(s-s')\rho(s')ds' \approx -\frac{W_r}{k_r^2}\frac{\partial}{\partial s}\rho(s)$$

Comparing this result with formula (1) we can find the effective inductance of the resonator wake function

$$L_r = \frac{1}{c^2}\frac{W_r}{k_r^2}$$

As an example we give the estimation for the inductance of a bellow:

$$L_b \approx \mu_0\frac{hl}{4\pi a}$$

where  $l$  is the length,  $a$  is the radius and  $h$  is the corrugation depth of a bellow.

## 3 QUASI GREEN FUNCTION

There are only few known analytical expressions for the wake (Green) function. Green function is the wake potential of a point charge for a particular accelerator element. Numerical solutions of the Maxwell equations can give wake potentials only for the bunches of finite length. There are some approximate methods of how can wake potential of a short bunch be used as a Green function. However the application of these methods is limited by this short

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bunch length. Besides, they can not restore the wake potential of this bunch. To overcome this problem we introduce a new approximate wake function  $\tilde{w}$ - "Quasi Green Function" - with additional distance parameter  $s_q$   $\tilde{w} = \tilde{w}(s_q, s)$  By means of this quasi Green function the approximation for the the wake potential  $\tilde{W}(s)$  is calculated directly by the integral convolution with the bunch density:

$$\tilde{W}(s) = \int_0^\infty \tilde{w}(s_q, s') \rho(s + s_q - s') ds' \quad (2)$$

It can be seen that this function approaches real Green function, when parameter  $s_q \rightarrow 0$ . Quasi Green function approaches the bunch wake potential  $\tilde{w}(s_q, s) \rightarrow \tilde{W}(s - s_q) \approx W(s - s_q)$ , (shifted in distance  $s_q$ ), when the parameter is larger than the bunch length  $s_q \gg \sigma_z$ . This quality gives the way for the evaluation of the quasi Green function from the wake potential. However we can try to find better way to derive this function. For example, by solving the problem on the extremum for the functional:

$$\min_{\tilde{w}} \int_{s_{min}(s_q)}^{s_{max}} \left( W(s) - \tilde{W}(s) \right)^2 ds \quad (3)$$

Optimum value for the parameter  $s_q$  is determined by the quality, and resolution of the basic wake potential  $W(s)$ , that was calculated from the Maxwell equations. We show how this method works on the example of the calculations of the wake potentials for a bellow (Fig. 2). We take the already calculated wake potential of the 1mm bunch and evaluate the quasi Green function by solving the problem (3) and reconstruct the wake potentials by (2) for the bunches of 46.6mm, 1.0mm and 0.5 mm length. One can

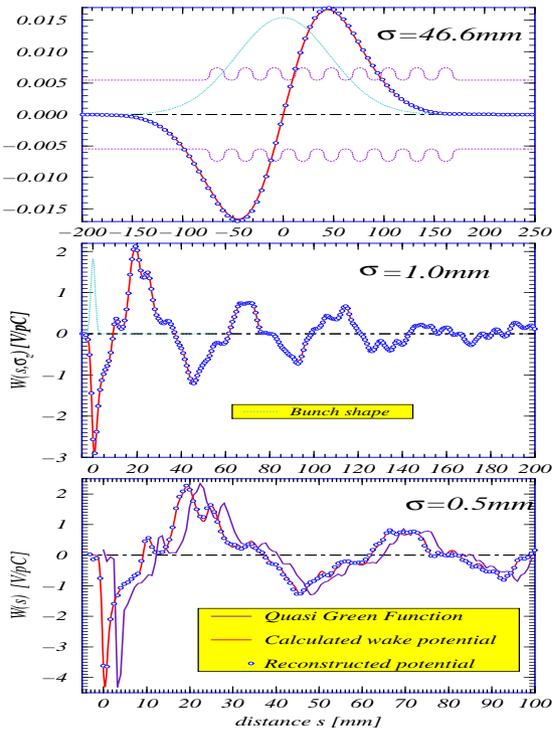


Figure 2: Wake potentials of a bellow (calculated and reconstructed) and Quasi Green Function.

see good agreement with the numerical calculations, even

for the wake potential of the twice as short bunch. The shape of the quasi Green function is not very far from the wake potential of 0.5mm bunch.

## 4 FOKKER-PLANCK EQUATION

To study the effect of the wake fields on the longitudinal beam dynamics in a storage ring, opposite to the usual way of multiparticle tracking [4],[5],[6],[7] we use the numerical solutions of the Fokker-Planck Equation. This equation describes the particle distribution function  $\psi = \psi(\tau, x, p)$  in the phase plane

of coordinate and momentum  $(x, p)$

$$\frac{\partial \psi}{\partial \tau} + \dot{x} \frac{\partial \psi}{\partial x} + \dot{p} \frac{\partial \psi}{\partial p} = \lambda \frac{\partial}{\partial p} (p\psi + \frac{\partial \psi}{\partial p}) \quad (4)$$

The coordinate and momentum are normalized by natural (zero-current) value of the bunch length  $\sigma_0$  and momentum spread  $\delta_0$ . Time is measured in synchrotron periods. Damping time is  $\tau_{damp} = \frac{2}{\lambda}$ . Time derivatives of the canonical coordinates are :  $\dot{x} = p$

$$\dot{p} = \frac{\sin(\varphi_0) - \sin(k_{rf}x + \varphi_0)}{k_{rf}} + \frac{eNc}{V_{rf}\omega_{rf}\sigma_0} \tilde{W}(\tau, x)$$

$q = eN$  is the bunch charge,  $V_{rf}$  is the RF voltage from the cavity,  $\omega_{rf}$  is the frequency of this RF. Wake potential  $\tilde{W}(\tau, x)$  is calculated in the presented above way (2) for the bunch density  $\rho(\tau, x)$

$$\rho(\tau, x) = \int_{-\infty}^{+\infty} \psi(\tau, x, p) dp$$

Additionally to the r.m.s. bunch length and momentum spread we calculate the power of the energy loss of the bunch  $\kappa(\tau)$

$$\kappa(\tau) = \int_{-\infty}^{+\infty} W(\tau, x) \rho(\tau, x) dx$$

For the solution of the Fokker-Planck equation we use numerical method, based on the original implicit finite-difference algorithm of fourth order. This method provides correct dispersion relation, up to the mesh size wavelength and does not produces any numerical diffusion, distortion or modulation. We checked the algorithm for the high-frequency resonator wake function ( $ks_0 = 25$ ) for the case of damped synchrotron oscillations ( to the stable solution of Haissinski equation ). We also made comparison with the results of multi-particle tracking simulations of sawtooth instability [6] for low-frequency field  $ks_0 = 0.5$ .

## 5 LONGITUDINAL BEAM DYNAMICS STUDY

The main parameters that determine the beam dynamic are: frequency of the resonator wake field  $k\sigma_0$  and intensity parameter  $I$

$$I = \frac{eNc}{V_{rf}\omega_{rf}\sigma_0} W_r$$

Varying this parameter we get different beam behavior.

Bunch lengthening. The bunch length is increased when momentum spread damps to the natural value and energy loss comes to constant value.

Weak instability. Small oscillations. Bunch length and momentum spread go up. Fig. 3 presents the result for  $I = 73$  and  $k\sigma_0 = 12.25$ . Bunch energy loss shows precisely the very small instability growth.

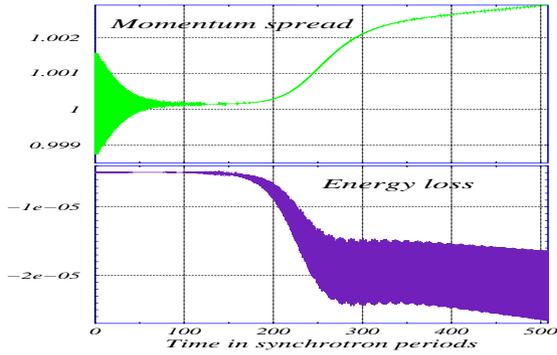


Figure 3: Weak instability.

Strong instability. Large oscillations of the bunch length and momentum spread. On Fig. 4 the upper curves show the emittance, bunch length and energy spread in time (in synchrotron periods). Left graphic shows the energy distribution and right - the particle distribution together with cavity RF voltage and wake potential. Below one can see the beam on the phase plane and its colored projection.

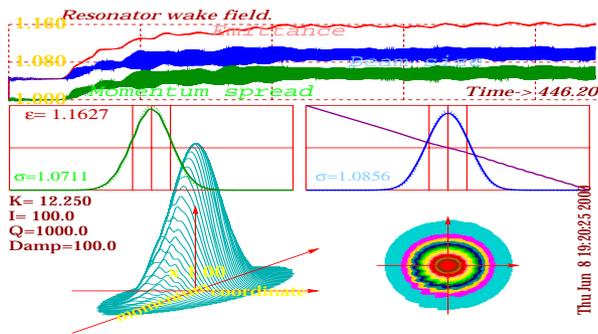


Figure 4: Strong instability

Saw-tooth Instability (Fig. 5). Adiabatic transition in the damping period and quick resonance microbunching in the blowup period. Metastable states or strange attractors (Fig. 6) and stochastic bursts of radiation (Fig. 7). The

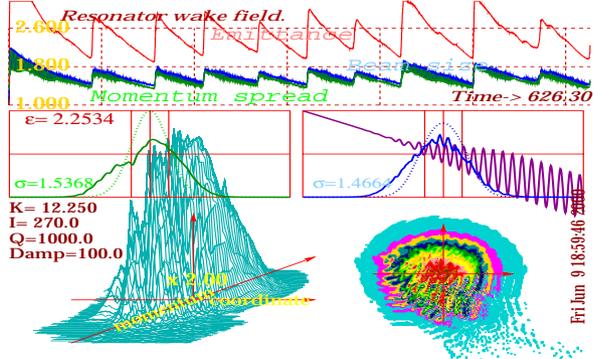


Figure 5: Saw-tooth instability

instability threshold was calculated in the large frequency band. The results are shown in Fig. 8. High frequency fields produce mainly saw-tooth instability. The estimation for threshold ( $I_{thr} \approx 1.8(k\sigma_0)^2$ ) can be evaluated from the

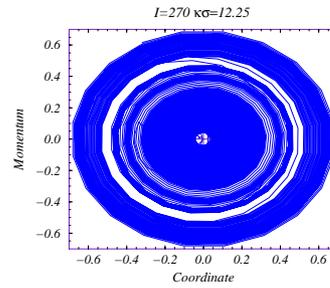


Figure 6: Bunch trajectory during 850 synchrotron periods.

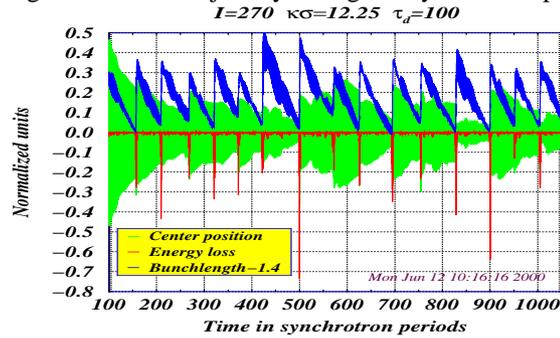


Figure 7: Bunch length, center position and energy loss.

assumption that instability begins when the wake energy spread becomes comparable with the RF focusing. We have found the coefficient for this relation from our simulations

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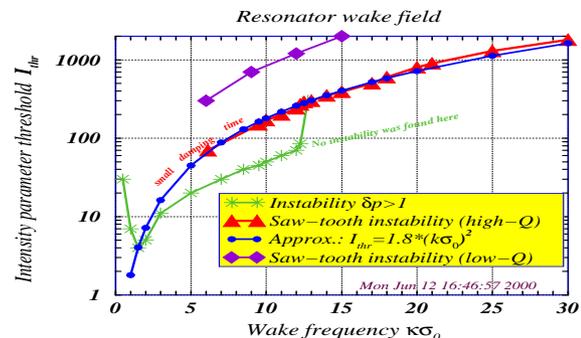


Figure 8: Instability threshold.