MATRIX REPRESENTATION FOR THE VLASOV-MAXWELL EQUATION SOLUTION

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Abstract

The evolution of high-intense particle beams is of great interest to many areas of modern applied science: accelerators for high-energy physics, free electron lasers, heavy-ion fusion drivers, spallation neutron sources, and medical applications, among others. In this paper we discuss for the transport of space-charge dominated beams. A new particle beam simulation approach based on the perturbation methods for distribution function is discussed. On the first step this approach compares the Vlasov-Maxwell equation system an infinite system of nonlinear differential equations. On the second step using the matrix formalism for Lie algebraic tools we build a linear algebraic equation system for aberration matrices for any approximation order in a symbolic form. Symbolic representation of distribution function allow to investigate beam dynamic more carefully. Some results of beam computing (including space charge forces) are demonstrated.

1 INTRODUCTION

The need for high current in a variety of new accelerator applications has focused an attention on effective and adequate methods of beam distribution describing including space-charge forces. In particular, the phenomenon of halo formation in high-intensity charged particle beams demands careful beam description, which can give correct results and formulate methods for halo control. In starting works (see, for example, [1]-[3]) two-dimensional simulation studies has led to the conclusion that halo is formed due to a mismatching of a beam to a focusing channel and may be explained by some sort of collective nonlinear oscillations of the beam. Most of the simulation studies start with rms matched beams with the help of the nonstationary Vlasov-Maxwell equations system ([4]). This allows to describe a redistribution of the initial beam phase portrait $\mathfrak{M}_0$ to a current one $\mathfrak{M} = \mathfrak{M}(s)$ and to study possible mechanism of halo formation. Most of works devoted to halo problem are based on the particle-core model. But they do not address the question of whether halo formation is influenced by the density redistribution which follows for a nonstationary beam. We suggest to study the halo development mechanism by virtue of the redistribution process. Such an approach allowed us to study mechanisms of halo formation associated with distribution forms and external fields characteristics. To accomplish this we present solutions of Vlasov-Maxwell equations using matrix representation for Lie algebraic tools [5]. Then we explore the redistribution process for beam phase portrait [6]. Such kind of simulation studies of beam distribution in an accelerator have provided useful physical insights into the dynamics of high-current beams. This is the focus of the present paper.

2 THE VLASOV-MAXWELL EQUATIONS SYSTEM

In this paper we model the transverse dynamics for beams where the bunch length is several times larger than its transversal sizes, and all our reasoning will be relate to transverse particle motion in the space-charge field of a continuous cylindrical beam with an elliptical section. It is known that for adequate treatment of beam evolution we should use beam description on the phase distribution function language.

Usually there are used two forms of beam distribution functions presentation. In the traditional form the distribution function is presented as a function of phase (transverse) coordinates: $f(X) = f(Q,P) = f(q_x, p_x, q_y, p_y)$, $Q = (q_x, q_y)^*$, $P = (p_x, p_y)^*$, $X = (Q,P)^*$. In the second form the distribution function is presented as a function of motion integrals for particle motion equations $dX/ds = F(X) = F(E,B,s)$. For example, one can write $f = \varphi(H)$, where H is a Hamiltonian of the dynamical system consisting of an external control system and a beam. The Vlasov equation can be written in the following form (s is a distance measured along a reference orbit of the machine)

$$\left\{\frac{\partial}{\partial s} + V\frac{\partial}{\partial Q} + F\left(E,B,s\right)\frac{\partial}{\partial P}\right\} f(Q,P,s) = 0, \qquad (1)$$

where $F(E,B,s)$ is the Lorenz force, written in the chosen coordinates system, and the particle velocity V and impulse P are connected with the relativistic relation:

$$V = \frac{P/m_0}{\sqrt{1 + (P,P)/m_0^2c^2}}.$$

Electromagnetic field and Hamiltonian can be decomposed in two parts:

$$E = E_{ext} + E_{self}, \quad B = B_{ext} + B_{self},$$

$$H = H_{ext} + \lambda\Phi_{self},$$

where E_{ext}, B_{ext}, H_{ext} are electric, magnetic fields and a Hamiltonian in absent of space charge forces, λ =

$qI/m_0c^3\beta^3\gamma^3\varepsilon_0$ – the Coulomb parameter, I – a beam current, m_0 , q – rest mass and charge of particles, β – dimensionless velocity of beam particles, γ – relativity factor, Φ_{self} – self-consistent potential. For every fixed value of s

$$\nabla^2\Phi_{self} = -\frac{2\pi K_0}{N_0}\int f(Q,P)dP, \qquad (2)$$

where K_0 and N_0, are the self-field perveance and the number of beam ions per unit axial length [7], respectively. The nonlinear Vlasov-Maxwell equations system (1) and (2) is used to study detailed beam propagation in the control system over a wide range of system parameters.

3 DISTRIBUTION FUNCTION EVOLUTION

3.1 Matrix Formalism for Lie Algebraic Methods

Following to previous papers [6], [8] we present the force function $F(E,B,s)$ in the following series

$$F\left(E(X,s),B(X,s),s\right) = \sum_{k=0}^{\infty}\mathbf{P}^{1k}(E_{\{k\}},B_{\{k\}},s)X^{[k]},$$

where $X^{[k]}$ is the Kronecker power of k-th order of phase vector X, $E_{\{k\}} = E_{\{k\}}(s)$, $B_{\{k\}} = B_{\{k\}}(s)$, are external field functional characteristics (for example, field distribution along the reference orbit). This expansion leads us to the following solution form for particle motion equations:

$$X(s) = \sum_{k=0}^{\infty}\mathbf{M}^{1k}\left(E_{\{k\}},B_{\{k\}},s\right)X_0^{[k]}, \qquad (3)$$

where $X_0^{[k]} = X^{[k]}(s_0)$ for some initial value of s_0, $\mathbf{M}^{1k}$ are aberration matrices of k-th order. We should note that the matrix formalism allows to computer these matrices in advance in a symbolic form (using, for example, the computer algebra codes MAPLE V). Similar representation for Lie transformation [8] allows us to investigate redistributing process for the beam phase portrait more carefully.

3.2 Distribution Function Presentation

As we mentioned above there are two forms of distribution function presentation: as a function of phase coordinates and as a function of motion integrals (for example, over Hamiltonian for stationary systems). The first form has also two forms: a form of an arbitrary function of X and a form of an arbitrary function of some special function of X (for example, a quadratic form $\kappa^2 = X^*\mathbf{A}X$, where $\mathbf{A}$ is a positive definite, symmetric matrix). In both cases we assume that these functions admit the power expansions in the forms:

$$f(X,s) = \sum_{k=0}^{\infty}\left(F_k(s)\right)^* X^{[k]}, \qquad (4)$$

or

$$f(X,s) = \varphi(\kappa^2(s)) = \sum_{k=0}^{\infty} f_k(s)\kappa^{[2k]}, \qquad (5)$$

where F_k and f_k vector and scalar coefficients respectively, and for $\kappa^{[2k]}$ we can write: $\kappa^{[2k]} = \left(X^{[k]}\right)^*\mathbf{A}^{\{k\}}X^{[k]}$, $\mathbf{A}^{\{k\}}$ is a symmetrized Kronecker power of $\mathbf{A}$ [8].

In the second case one can write

$$f(Q,P,s) = \Psi(I_1,\ldots,I_m), \qquad (6)$$

where $I_1,\ldots,I_m)$ are motion integrals. According to the perturbation approach we can present a motion integral in the form similar to (4) or (5) too (see [9]).

3.3 Beam Portrait Propagator

Particle motion equations generate a map $\mathcal{M}(F;s|s_0)$: $\mathfrak{M}_0 \mapsto \mathfrak{M}$ called Lie transformation [10]. Using its properties we can write for a current image distribution function $f(X,s)$ the following relation

$$f(X,s) = f_0\left(\mathcal{M}^{-1}\circ X\right). \qquad (7)$$

The matrix representation for Lie transformation $\mathcal{M}$ in the Poincare-Witt basis (it can presented in the form: $\left\{1\ X\ X^{[2]}\ \ldots X^{[k]}\ldots\right\}$) has the form (3). For the first form of distribution function presentation (4) we can write

$$f(X,s) = f_0(\mathcal{M}^{-1}\circ X_0) = \sum_{k=0}^{\infty}\left(F_k(s)\right)^*\left(\mathcal{M}^{-1}\circ X_0\right)^{[k]} =$$

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{k}\left(F_l(s)\right)^*\mathbf{T}^{kl}\left(s|s_0\right)X_0^{[k]}, \qquad (8)$$

where $\mathbf{T}^{kl}(s|s_0)$ are block matrices corresponding to the inverse Lie transformation $T = \mathcal{M}^{-1}$, and they can be easily calculated using the generalized Gauss algorithm. Similar formula can be written for the second form of distribution functions (5) too. We also use a presentation of a phase portrait with the help of level surfaces. Indeed, using reciprocal locations of these surfaces (curves on a plane) one can model the initial distribution function:

$$\left(f_0(X) = c_\alpha,\ \ \alpha = \overline{1,N}\right) \Longleftrightarrow \left(G_0(X) = C\right)$$

where C is a vector of constants $C = (\tilde{c}_1,\ldots,\tilde{c}_N)^*$, $\tilde{c}_k = \tilde{c}_k(c_k)$ and N is a number of level surfaces. According to the Liouville theorem the image of these surfaces describe with the following relations

$$G(X,s) = G_0(\mathcal{M}^{-1}\circ X_0) = C.$$

Using such presentation for beam distribution functions we can study redistribution process for beam particles in different transport and focusing channels. Moreover, this approach allows to formulate optimal criteria for different kind of beam lines in terms of conditions for matrices elements (corresponding conditions take the form of algebraic equations for these elements). For example, if one wants to obtain an uniform distribution at some section along the beam line (or at a target) he should demand fulfilment of the following conditions up to some approximation order M (here $\hat{s}$ is a section location):

$$\sum_{l=1}^{k} \left(F_l(\hat{s})\right)^* \mathbf{T}^{kl}\left(\hat{s}|s_0\right) = 0, \quad \forall\, 1 \le k \le M. \tag{9}$$

The other direction of our investigations lays in the field of space charge effects, in particular, for halo formation problems. The symbolic representation of the model distribution function both in the phase and configuration spaces allows to analyze an influence of different parts of the particle set $\mathfrak{M}_0$ on the halo formation process.

4 GRAPHIC EXAMPLES

We should note that suggested approach naturally admits usage of computer algebra methods and tools. As examples of this we include to this work some demonstrating pictures of halo formation process both in phase space $\{r, p_r\}$ and in configuration space $\{x, y\}$ (see the Fig.1 and the Fig.2). Here the following Gaussian-like distribution is studied:

$$f_0(X) = \varphi(\kappa_0^2) = P_n(\kappa_0^2) \exp\left(-\frac{Q_m(\kappa_0^2)}{2\sigma^2}\right),$$

where $\kappa_0^2 = X^* \mathbf{A}_0 X$ is a quadratic form describing an phase ellipsoid occupied by beam particles at the initial moment s_0, $P_n(t)$, and $Q_m(t)$ are polynomials on a variable t of n, m degrees respectively. The forms of the polynomials P_n and Q_m were varied. The goal of this process is to study of influence of so called tail particle [8]. We should note that this approach allows to investigate what part of initial distribution contributes to the central part (core) and surrounding part (halo). Using discussed approach we compute evolution different part of the phase beam portrait and show that the core emits particles which pass to halo step by step. On the other hand there are so called tail particles keeping this properties and after halo formation remain there.

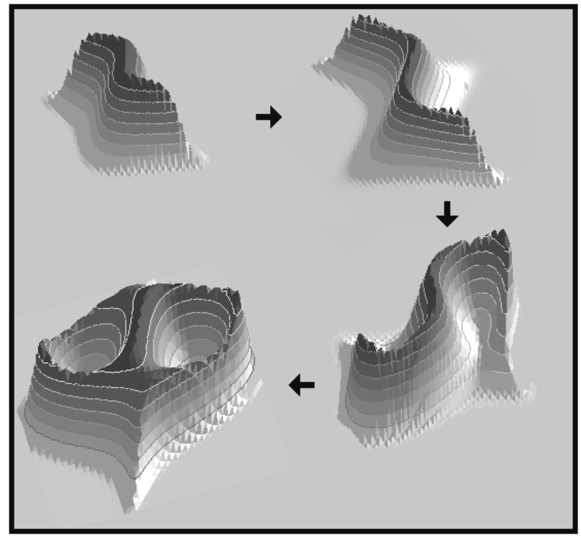

Figure 1: Halo formation for proton beam with a Gaussian-like distribution in a nonlinear focusing channel up to third order aberration. The sequence of beam phase portraits in the $\{r, r'\}$ plane is showed by arrows.)

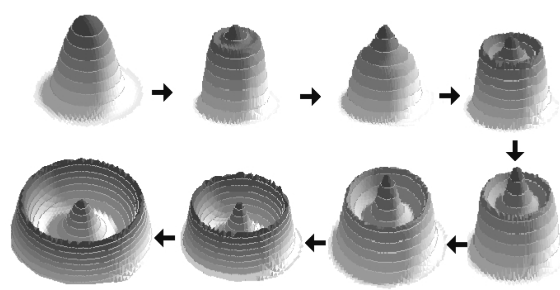

Figure 2: Halo formation process for proton beam with a Gaussian-like distribution in a nonlinear focusing channel up to third order aberration. The sequence of beam spots in the $\{x, y\}$ plane is showed by arrows.

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