TURN-BY-TURN PHASE SPACE DIAGRAM CONSTRUCTION FOR NON-LINEAR BETATRON OSCILLATION

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Abstract

The problem of phase space diagram construction for non-linear betatron oscillation measured by pickup, is considered. The conventional two-pickup method [1] of phase trajectory construction was improved. Discrete Fourier filter applied to data measured yields a large dividend in accuracy. The result of our investigations is the method of turn-by-turn phase trajectory construction using data measured by single pickup. The single-pickup method developed was tested by computer simulation of non-linear betatron oscillation in several models of magnet lattice. Practicality of the method and its accuracy limitation were studied. The method applying for experimental study of beam dynamic is discussed.

1 INTRODUCTION

Phase space diagram of non-linear betatron oscillation gives a lot of information about the non-linearity type, non-linear resonances, dynamic aperture, etc. It is useful to compare phase trajectories of the beam motion measured with results of analytical estimations and numerical simulations.

A problem of phase trajectory construction is obtaining $x'(x)$ dependence, where $x(t)$ is transverse coordinate, and $x'(t)$ is transverse momentum of a beam centre of charge, t is a time variable. The problem is troublesome because of impossibility to measure directly the momentum $x'(t)$.

Diagnostic systems give an information about beam motion as series of turn-by-turn samples x_k of the coordinate measured by pickup. Due to the discreteness, calculation of the momentum samples x'_k, by numerical differentiation of the x_k, is impossible in general.

Let's consider a problem of construction of turn-by-turn phase trajectory $x'_k(x_k)$ of non-linear betatron oscillation using the coordinate samples x_k.

It is convenient to analyse phase trajectories in the $(x,\ \bar{x}')$ coordinates defined by the variables conversion:

$$\bar{x}' = \alpha x + \beta x'. \tag{1}$$

A shape of phase trajectory in these coordinates is independent of the value of alpha-function $\alpha = -\beta'/2$, but this shape is determined by pure non-linear effect.

2 TWO-PICKUP METHOD

There are conventional two-pickup method [1] of turn-by-turn phase trajectory construction. Let's consider particle motion in a linear section of magnet lattice with two pickups, first of them placed at the input of the section and second at the output of it.

If a particle passes through the section, its coordinate x_2 measured by the second pickup is:

$$x_2 = (\beta_2/\beta_1)^{1/2}\cdot(x_1 \cos\Delta\psi_{21} + \bar{x}'_1\cdot\sin\Delta\psi_{21}), \tag{2}$$

here x_1 is the coordinate and $\bar{x}'_1$ is the normalized momentum at the first pickup, $\beta_{1,2}$ are the values of beta-function at the pickups, $\Delta\psi_{21}$ is the betatron phase advance between the pickups. From this expression an equation of turn-by-turn phase trajectory is derived:

$$\bar{x}'_{1k} = [(\beta_2/\beta_1)^{1/2}\cdot x_{2k} - x_{1k}\cos\Delta\psi_{21}]/\sin\Delta\psi_{21}\ . \tag{3}$$

If $\beta_1 = \beta_2$ and $\Delta\psi_{21} = \pi/2+\pi n$, then $\bar{x}'_{1k} = x_{2k}$.

An accuracy of the method is determined by pickup resolution in the frequency band with upper bound equal to the revolution frequency. The noise error leads to poor quality of phase trajectories constructed by this method.

For decrease the noise error, a method of discrete Fourier filtering was developed. Let's expand the arrays x_{1k} and x_{2k} of N turn-by-turn coordinate samples in terms of harmonics $\Phi_{1,2\,j}=A_{1,2\,j}+iB_{1,2\,j}$ of betatron frequency Q:

$$A_{1,2\,j} = 2/N\cdot\sum_{k=0}^{N-1} x_{1,2\,k}\cdot\cos(2\pi k\cdot jQ) \tag{4}$$
$$B_{1,2\,j} = 2/N\cdot\sum_{k=0}^{N-1} x_{1,2\,k}\cdot\sin(2\pi k\cdot jQ)$$

Amplitude of harmonics $|\Phi_{1,2\,j}| = (A_{1,2\,j}^{\ 2}+B_{1,2\,j}^{\ 2})^{1/2}$ in (4) decreases rapidly with the harmonic number j.

Procedure of turn-by-turn phase trajectory construction is just the synthesis of the arrays X_{1k}, X_{2k}:

$$X_{1,2\,k} = \sum_{j=1}^{n} (\,A_{1,2\,j}\cdot\cos 2\pi kjQ + B_{1,2\,j}\cdot\sin 2\pi kjQ\,) \tag{5}$$

The $X_{2k}(X_{1k})$ dependence describes the phase trajectory.

Noise component of the j–th harmonic in (5) is $N^{1/2}$ times lower than broad-band noise component of the x_{1k}, x_{2k} arrays. If the number of harmonics in (5) $n << N$, then noise reduction is $(N/n)^{1/2}$. So, combination of the expansion (4) with the synthesis (5) is a discrete filter. Usually $N = 1024$, $n = 4{\div}8$, so typical noise reduction by the filter is 10÷15 times.

An example of the filter applying to the two-pickup method is given in Fig. 1. There are phase trajectories of radial betatron oscillation in the VEPP-4M near the sextupole resonance $3Q_x = 26$. The trajectory $x_{2k}(x_{1k})$, constructed by the two-pickup method without filtering, is plotted by circles, the trajectory $X_{2k}(X_{1k})$ constructed using the filter is plotted by triangles.

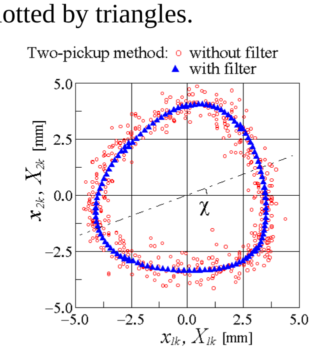


Figure 1: Applying of the discrete Fourier filter.

Broad-band noise resolution of the turn-by-turn pickup is about 100 μm, the filter decreases noise error down to ~10 μm.

3 SINGLE-PICKUP METHOD

There is a limitation of practicality of the two-pickup method, imposed by non-linear field components of the elements placed between the pickups. In presence of the components, coordinate transform is not described by (2).

Thus, if a magnet lattice has no linear section with betatron phase advance of the order of $\pi/2$, the two-pickup method had failed. In this case the problem of phase trajectory construction using coordinate samples measured by single pickup, is of a special interest.

Let's analyse a particle motion in the two utmost models of non-linear lattice: the lattice with a uniform distribution of non-linearity and the lattice with a single non-linear element.

3.1 Uniformly Distributed Non-linearity

Equation of particle motion in a magnet lattice with a uniform distribution of non-linearity is:

$$x'' + \Omega^2 x = f_n \cdot x^n, \qquad (6)$$

here $\Omega^2 = K_x$ is focusing, f_n is n-th order multipole coefficient of non-linear force.

For $\Omega^2 = \text{const}$ (azimuthal symmetric field), this equation can be solved analytically, solution has a form of $x'(x)$ and describes a phase trajectory of the motion. For sextupole non-linearity ($n = 2$) turn-by-turn relation between momentum x'_k and coordinate x_k is:

$$x'_k = \pm\Omega^{-1} \cdot (C - x_k^2 \pm 2/3 \cdot \Omega^{-2} \cdot f_n \cdot x_k^3)^{1/2}. \qquad (7)$$

The coefficients f_n and C can be obtained by analysis of the array x_k of turn-by-turn coordinate samples.

Thus, for a magnet lattice with a uniform distribution of non-linearity, there are turn-by-turn relations between x'_k and x_k independent of non-linearity magnitude and oscillation amplitude.

3.2 Single Non-linear Element

Equation of particle motion in a magnet lattice with a single non-linear element is:

$$x'' + \Omega^2 x = \sum_{k=0}^{\infty} f(x) \cdot \delta(\theta - \Delta\theta + 2\pi k), \qquad (8)$$

here $\Omega^2 = K_x$ is focusing, non-linear element placed at the $\Delta\theta$ azimuth is modeled by the product of non-linear function $f(x)$ by delta-function $\delta(\theta - \Delta\theta + 2\pi k)$ "switching on" non-linear force at each turn.

This equation can be solved analytically at each turn using Laplace transform. Turn-by-turn samples of coordinate x_k and momentum x'_k are:

$$x_k = x_0 \cos 2\pi kQ + \beta x'_0 \cdot \sin 2\pi kQ - \beta \cdot \sum_{m=0}^{k-1} f_m \cdot \sin[2\pi Q(k-m) - \Delta\psi], \qquad (9)$$

$$x'_k = -1/\beta \cdot x_0 \sin 2\pi kQ + x'_0 \cdot \cos 2\pi kQ - \sum_{m=0}^{k-1} f_m \cdot \cos[2\pi Q(k-m) - \Delta\psi], \qquad (10)$$

here Q is betatron frequency, $\Delta\psi = Q\Delta\theta$ is betatron phase advance between non-linear element and pickup.

From the expressions (9), (10) for k-th and (k+1)-st turns, a recurrent formula to calculate the momentum x'_{k+1} is derived:

$$x'_{k+1} = [x_{k+1} \cdot \cos(2\pi Q - \Delta\psi) - x_k \cdot \cos\Delta\psi - \beta \cdot x'_k \cdot \sin\Delta\psi]/\beta \sin(2\pi Q - \Delta\psi), \qquad (11)$$

It is remarkable that the non-linear force f_k at each turn can be calculated by the formula:

$$f_k = (x_k \cos 2\pi Q - x_{k+1} + \beta x'_k \cdot \sin 2\pi Q)/\beta \sin(2\pi Q - \Delta\psi), \qquad (12)$$

Sorting $f_k(x_k)$ by increase of x_k, one can approximate the function $f(x)$ and determine type of the non-linearity.

Note, that the phase trajectory constructed for $\Delta\psi \neq 0$ is transformed by rotation on the $-\Delta\psi$ angle to the phase trajectory constructed for $\Delta\psi = 0$:

$$x'_{k+1} = (x_{k+1} \cdot \cos 2\pi Q - x_k)/\beta \sin 2\pi Q, \qquad (13)$$

Thus, for a magnet lattice with a single non-linear element, there is a recurrent formula (11) to calculate the turn-by-turn momentum x'_k from the coordinate x_k.

Analysis of these two utmost cases gives promise that for some distributions of non-linear lattice elements there are relations between x'_k and x_k independent of non-linearity magnitude and oscillation amplitude.

3.3 Amplitude-independent relations between coordinate and momentum spectra

As it was clarified, turn-by-turn samples of coordinate x_k and momentum x'_k are related to each other. This suggests that relations independent of non-linearity magnitude and oscillation amplitude are valid between coordinate Φ_j and momentum Φ'_j spectra.

An expansion of x_k array of N samples in terms of harmonics $\Phi_j = A_j + iB_j$ of betatron frequency Q is:

$$A_j = 2/N \cdot \sum_{k=0}^{N-1} x_k \cdot \cos(2\pi k \cdot jQ), \qquad (14)$$
$$B_j = 2/N \cdot \sum_{k=0}^{N-1} x_k \cdot \sin(2\pi k \cdot jQ).$$

For the model lattice with a single non-linear element, frequency depended expressions for the relations between momentum harmonics $\Phi'_j = A'_j + iB'_j$ and coordinate ones $\Phi_j = A_j + iB_j$ are derived from the recurrent formula (11) using the harmonic expansion (14), with neglect of the terms of the order $1/N$.

$$A'_j = [A_j (\cos 2\pi Q - \cos 2\pi jQ) - B_j \cdot \sin 2\pi jQ]/\beta \sin 2\pi Q, \qquad (15)$$
$$B'_j = [A_j \cdot \sin 2\pi jQ - B_j (\cos 2\pi Q - \cos 2\pi jQ)]/\beta \sin 2\pi Q,$$
$$j = 1, 2, \ldots, N \qquad Q \neq 0, 0.5, 1, \ldots$$

For the lattice with uniformly distributed non-linearity, there are simple expressions for the relations between normalized amplitudes a'_j/a'_1 and phases φ'_j of momentum harmonics and a_j/a_1, φ_j of coordinate ones:

$$a'_j/a'_1 = j \cdot a_j/a_1, \qquad \varphi_j - \varphi'_j = \pi/2, \qquad (16)$$

Note, that the amplitude-phase relations (16) are independent of betatron frequency Q unlike the (15).

Non-linear oscillation in several types of magnet lattice was studied by computer simulation. One more example of such the relations using is presented in [2]. The amplitude-phase relations empirically obtained were used for phase trajectory construction at LEP:

$$a'_j/a'_1 = a_j/a_1, \qquad \varphi_j - \varphi'_j = \pi/2, \qquad (17)$$

Thus, if non-linear oscillation can be described by equation of motion, the amplitude-phase relations can be tabulated by analytical or numerical solution of the equation.

In general, amplitude-phase relations between coordinate and momentum spectra can be obtained in one way or another. These relations are independent of non-linearity magnitude and oscillation amplitude and can be use for turn-by-turn phase trajectory construction.

4 PRACTICAL USE OF THE METHOD

The formulas (11), (15) obtained by analysis of the simple model lattice with a single non-linear element can be used for study of non-linear betatron oscillation in real accelerators. This model approximately describes a motion in accelerator with low-order symmetry, lattice of which has final focus. In this case sextupole chromaticity correctors are placed close to final focus quadrupoles where beta functions is large (at VEPP-4M — more than 10 times greater than the mean value), and these sextupole correctors are dominated in the non-linearity.

Practicality of the single-pickup method was tested using computer simulation. Phase trajectories constructed by the method were compared with results of particle tracking in the VEPP-4M lattice with sextupoles.

As an accuracy criterion of the method, the correlation coefficient between the phase trajectory constructed and the phase trajectory calculated by computer tracking, was used. A value of the coefficient close to 1 attests that the phase trajectories are close to one another, and the method accuracy is rather good. It was discovered that in the practically interesting range 8.62÷8.75 of the VEPP-4M radial betatron frequency, the correlation coefficient is more than 0.9.

The single-pickup method was tested also by comparison with the conventional two-pickup method. Example of the phase trajectory constructed by these methods is shown in Fig. 2. The trajectory plotted by circles was constructed by the two-pickup method with Fourier filtering, the trajectory plotted by triangles was constructed by the single-pickup method.

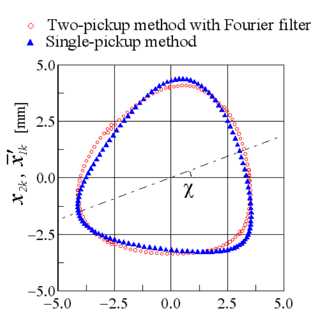
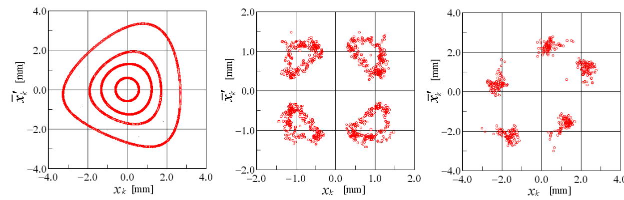


Figure 2: Two-pickup and single-pickup methods.

The single-pickup method was used for experimental study of non-linear beam dynamics at the VEPP-4M [3]. In Fig. 3 examples of the phase trajectories constructed by the single-pickup method are presented. Fig. 3*a* illustrates betatron oscillation of varied amplitude near the $3Q_x$=26 resonance, Fig. 3*b* and Fig. 3*c* demonstrates non-linear resonances $4Q_x$=35 and $5Q_x$=43 respectively.



a) *b*) *c*)

Figure 3: Examples of phase trajectories measured.

The single-pickup method can also be used for construction of phase trajectory of synchrotron oscillations. Synchrotron oscillation produces oscillation of radial coordinate, turn-by-turn samples of which are:

$$x_k(\theta) = R\cdot\psi(\theta)\cdot(\Delta E/E)_k\cdot\exp[i(2\pi k\Omega/\omega_0+\chi)], \quad (18)$$

here $\psi(\theta)$ is dispersion function, $\Delta E/E$ is an energy deviation, proportional to time derivation of the synchrotron phase ϕ:

$$\Delta E/E = 1/q\omega_0 K_s\cdot d\phi/dt. \quad (19)$$

Phase trajectory of synchrotron oscillation can be constructed using the amplitude-phase relations:

$$a_{\phi_j}/a_{\phi_1} = (a_{E_j}/a_{E_1})\cdot j^{-1}, \qquad \varphi_{\phi_j} - \varphi_{E_j} = -\pi/2. \quad (20)$$

Fig. 4 shows the phase trajectory of synchrotron oscillation constructed from data measured in comparison with the result of computer simulation.

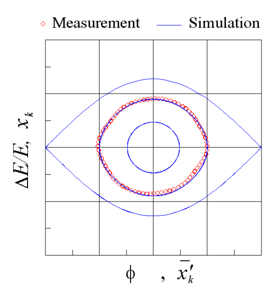


Figure 4: Phase trajectory of synchrotron oscillation.

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