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EXPLICIT EXPRESSIONS FOR NON-MAGNETIZED BUNCHED ELECTRON COOLING*

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Abstract

Recent success of Low Energy RHIC Electron Cooler (LEReC) leads the way in development of high energy electron coolers based on non-magnetized electron bunches accelerated by RF cavities. In this paper we derive explicit formulas for the friction force and the cooling rates in non-magnetized electron coolers in the presence of re-distribution of cooling decrements. We further consider several particular cases reducing the general expressions to simple analytic formulas useful for optimization of coolers' parameters.

INTRODUCTION

Redistribution of cooling between longitudinal and transverse directions [1] requires two conditions. The first one is a coupling between the longitudinal and transverse (in this paper we will consider the horizontal one) motion of an ion. This is created by the ions' dispersion in the cooling section (CS). The second condition is dependence of the longitudinal friction force on the horizontal position of an ion in the cooling section, i.e. the longitudinal component of the cooling force must have the transverse gradient. A robust way to create the required gradient (which will be the focus of this paper) is to introduce the electron beam dispersion in the CS.

In the following section we will derive the explicit expressions for the dynamical friction force in the presence of electron dispersion. Next, we will show how introduction of ion dispersion results in $x-z$ redistribution of the cooling rates. Finally, we will apply the obtained formulas to several "asymptotic" cases.

DYNAMICAL FRICTION FORCE

The general expression for the dynamical friction force in non-magnetized cooling is:

$$\vec{F} = -\frac{4\pi N_e e^4 Z_i^2}{m_e}\int \Lambda_C \frac{\vec{v_i}-\vec{v_e}}{|\vec{v_i}-\vec{v_e}|^3} f_e(r_e, v_e) d^3 v_e, \quad (1)$$

where N_e is the number of electrons per bunch, e and m_e are an electron's charge and mass, Z_i is an ion's charge number, v_e and v_i are electron and ion velocities, Λ_C is a Coulomb logarithm, which is a weak function of velocity and can be taken out of integral, and $f_e(r_e, v_e)$ is the electrons 6-D distribution function.

Assuming Gaussian electron distribution, in the presence of electron dispersion (D_e) in the cooling section and using $x - D_e\delta_e$ for horizontal electron coordinate (here $\delta_e = \frac{v_{ze}}{\beta c}$), one can write f_e in the form $f_e = \rho_e f_{ve}$, where

$$\rho_e = \frac{1}{\gamma(2\pi)^{3/2}\sigma_{1xe}\sigma_{ye}\sigma_{ze}} \exp\left(-\frac{x^2}{2\sigma_{1xe}^2} - \frac{y^2}{2\sigma_{ye}^2} - \frac{z^2}{2\sigma_{ze}^2}\right), \quad (2)$$

$$f_{ve} = \frac{1}{\gamma(2\pi)^{3/2}\sigma_{vxe}\sigma_{vye}\sigma_{1vze}} \times \\ \times \exp\left(-\frac{v_{xe}^2}{2\sigma_{vxe}^2} - \frac{v_{ye}^2}{2\sigma_{vye}^2} - \frac{(v_{ze}-\mu)^2}{2\sigma_{1vze}^2}\right), \quad (3)$$

with $\sigma_{1xe} = \sqrt{\sigma_{xe}^2 + D_e^2\sigma_{\delta e}^2}$, $\sigma_{1vze} = \sigma_{vze}\frac{\sigma_{xe}}{\sqrt{\sigma_{xe}^2+D_e^2\sigma_{\delta e}^2}}$ and $\mu = x\sigma_{vze}\frac{D_e\sigma_{\delta e}}{\sigma_{xe}^2+D_e^2\sigma_{\delta e}^2}$.

Now, we can rewrite Eq. (1) as:

$$\vec{F} = -C_0\rho_e \int \frac{\vec{v_i}-\vec{v_e}}{|\vec{v_i}-\vec{v_e}|^3} f_{ve} d^3 v_e\ , \quad (4)$$

where $C_0 = \frac{4\pi N_e e^4 Z_i^2}{m_e}\Lambda_C$. We further introduce an effective potential in the velocity-space:

$$U = C_0\rho_e \int \frac{f_{ve}}{|\vec{v_i}-\vec{v_e}|} d^3 v_e\ . \quad (5)$$

Noticing that components of the friction force can be presented by $F_{x,y,z} = \partial U/\partial v_{xi,yi,zi}$, one can reduce Eq. (4) to 1-D integrals [2, 3]. For the sake of clarity, we will consider the case of $\sigma_{vxe} = \sigma_{vye} \equiv \sigma_{v\perp e}$ (for detailed derivations and more general cases see [2-4]).

After some algebraic manipulations we get Binney's formulas [5] for friction force components:

$$\begin{aligned} F_{x,y} &= -C_1\rho_e v_{xi,yi}\int_0^\infty g_\perp(q)dq \\ F_z &= -C_1\rho_e(v_{zi}-\mu)\int_0^\infty g_z(q)dq \\ g_\perp(q) &= \frac{E(q)}{\sigma_{v\perp e}^2(1+q)^2\sqrt{\sigma_{v\perp e}^2 q+\sigma_{1vze}^2}}\ , \\ g_z &= \frac{E(q)}{(1+q)(\sigma_{v\perp e}^2 q+\sigma_{1vze}^2)^{3/2}} \\ E(q) &= \exp\left[-\frac{v_{xi}^2+v_{yi}^2}{2\sigma_{v\perp e}^2(1+q)} - \frac{(v_{zi}-\mu)^2}{2(\sigma_{v\perp e}^2 q+\sigma_{1vze}^2)}\right] \end{aligned} \quad (6)$$

where $C_1 = 2\sqrt{2\pi}N_e r_e^2 m_e c^4 Z_i^2\Lambda_C$.

To simplify final expressions we will further consider an approximation of small amplitudes ($v_i < \sigma_{ve}$). Then, one can analytically take integrals in Eq. (6). Switching to laboratory frame values $\sigma_{\delta e} = \frac{\sigma_{vze}}{\beta c}$ and $\sigma_{\theta e} = \frac{\sigma_{v\perp e}}{\gamma\beta c}$, we get:

$$\begin{aligned} F_x &= -C_2\rho_e h v_{xi}\Phi\left(\frac{\sigma_{\delta e}}{\gamma\sigma_{\theta e}}\frac{1}{h}\right) \\ F_z &= -2C_2\rho_e h(v_{zi}-Kx_i)\left[1-\Phi\left(\frac{\sigma_{\delta e}}{\gamma\sigma_{\theta e}}\frac{1}{h}\right)\right] \end{aligned}, \quad (7)$$

where $C_2 = \frac{2\sqrt{2\pi}N_e r_e^2 m_e c Z_i^2\Lambda_C}{\gamma^2\beta^3\sigma_{\theta e}^2\sigma_{\delta e}}$, $h = \frac{\sqrt{\sigma_{xe}^2+D_e^2\sigma_{\delta e}^2}}{\sigma_{xe}}$ and parameter $K = \beta c\frac{D_e\sigma_{\delta e}^2}{\sigma_{xe}^2+D_e^2\sigma_{\delta e}^2}$. The function Φ is given by:

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$$\Phi(d) = \begin{cases} \frac{d}{1-d^2}\left(\frac{\arccos(d)}{\sqrt{1-d^2}} - d\right), \ d < 1 \\ \frac{2}{3}, \ d = 1 \\ \frac{d}{d^2-1}\left(\frac{\log\left(d-\sqrt{d^2-1}\right)}{\sqrt{d^2-1}} + d\right), \ d > 1 \end{cases} . \quad (8)$$

The plot in Fig. 1 shows that Φ is a rather strong function of the ratio of electrons effective longitudinal and transverse velocity spreads in the range most relevant to bunched electron cooling.

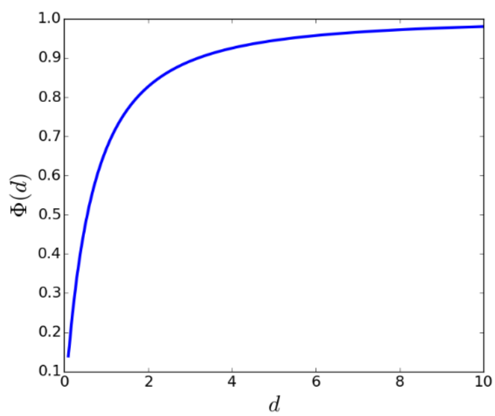

Figure 1: Function Φ.

COOLING REDISTRIBUTION

It follows from Eqs. (7) and (8) that on a single pass through the cooling section changes in an ion's angle ($\Delta x_i{}'$) and relative momentum ($\Delta\delta_i$) are given by:

$$\begin{aligned} \Delta x_i' &= c_0 \rho_e h x_{iCS}' \Phi\left(\frac{\sigma_{\delta e}}{\gamma\sigma_{\theta e}}\frac{1}{h}\right) \\ \Delta\delta_i &= 2c_0\rho_e h(\delta_{iCS} - kx_{iCS})\left[1 - \Phi\left(\frac{\sigma_{\delta e}}{\gamma\sigma_{\theta e}}\frac{1}{h}\right)\right] \end{aligned}, \quad (9)$$

where $c_0 = \frac{C_2 L_{CS}}{\gamma^2\beta A_i m_p}$, $k = \frac{K}{\beta c}$, L_{CS} is the length of the cooling section, A_i is an ion's atomic number and $x_{iCS}, x_{iCS}', \delta_{iCS}$ are ion's position, angle and relative momentum in the cooling section.

Denoting ion's parameters upstream and downstream of the cooling section with indexes 0 and 1 respectively and noticing that $x_{iCS} = x_{i0} + D_i\delta_{i0}$ and $x_{i1} = x_{iCS} - D_i(\delta_{i0} + \Delta\delta_i)$, we get $\Delta x_i = -D_i\Delta\delta_i$. Combining this with Eq. (9) we obtain:

$$\begin{aligned} \Delta x_i &= 2c_0\rho_{e1} h\left[1 - \Phi\left(\frac{\sigma_{\delta e}}{\gamma\sigma_{\theta e}}\frac{1}{h}\right)\right] D_i\left(\delta_{i0} - k(x_{i0} + D_i\delta_{i0})\right) \\ \Delta x_i' &= -c_0\rho_{e1} h \Phi\left(\frac{\sigma_{\delta e}}{\gamma\sigma_{\theta e}}\frac{1}{h}\right) x_{i0}' \\ \Delta\delta_i &= -2c_0\rho_{e1} h\left[1 - \Phi\left(\frac{\sigma_{\delta e}}{\gamma\sigma_{\theta e}}\frac{1}{h}\right)\right]\left(\delta_{i0} - k(x_{i0} + D_i\delta_{i0})\right), \end{aligned} \quad (10)$$

where the electrons density "probed" by an ion is:

$$\rho_e = \frac{1}{\gamma(2\pi)^{3/2}\sigma_{1xe}\sigma_{ye}\sigma_{ze}} \exp\left(-\frac{(x_{i0}+D_i\delta_{i0})^2}{2\sigma_{1xe}^2} - \frac{y_{i0}^2}{2\sigma_{ye}^2} - \frac{z_{i0}^2}{2\sigma_{ze}^2}\right). \quad (11)$$

We introduce horizontal and longitudinal actions of an ion, assuming that in the CS $\alpha_x = 0$:

$$\begin{aligned} J_x &= \frac{1}{2}\left(\frac{x_i^2}{\beta_x} + \beta_x x_i'^2\right) \\ J_z &= \frac{1}{2}\left(\frac{z_i^2}{\beta_z} + \beta_z \delta_i^2\right) \end{aligned}, \quad (12)$$

where $\beta_z \equiv \sigma_{zi}/\sigma_{\delta i}$.

The change in actions on a single pass through the CS is:

$$\begin{aligned} \Delta J_x &\approx \frac{x_{i0}\Delta x_i}{\beta_x} + \beta_x x_{i0}'\Delta x_i' \\ \Delta J_z &\approx \beta_z \delta_{i0}\Delta\delta_i \end{aligned}. \quad (13)$$

Finally, the change of i-bunch emittances over a single pass through the cooling section can be found as:

$$\begin{aligned} \Delta\varepsilon_x &= \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} f_i \Delta J_x dx_i \ldots d\delta_i \\ \Delta\varepsilon_z &= \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} f_i \Delta J_z dx_i \ldots d\delta_i \end{aligned}. \quad (14)$$

We assume the 6-D Gaussian distribution of ions:

$$f_i = \frac{\exp\left(-\frac{x^2}{2\sigma_{xi}^2} - \frac{y^2}{2\sigma_{yi}^2} - \frac{z^2}{2\sigma_{zi}^2} - \frac{x'^2}{2\sigma_{\theta xi}^2} - \frac{y'^2}{2\sigma_{\theta yi}^2} - \frac{\delta^2}{2\sigma_{\delta i}^2}\right)}{(2\pi)^3\sigma_{xi}\sigma_{yi}\sigma_{zi}\sigma_{\theta xi}\sigma_{\theta yi}\sigma_{\delta i}}. \quad (15)$$

Noticing that the cooling rate is $\lambda = \frac{1}{T_{rev}}\frac{\Delta\varepsilon}{\varepsilon}$, where T_{rev} is a revolution period in the ion storage ring, we get from Eqs. (10)-(15):

$$\begin{aligned} \lambda_x &= -P\left(c_x + c_z\frac{D_i^2\sigma_{\delta i}^2 + D_e D_i \sigma_{\delta e}^2}{D_i^2\sigma_{\delta i}^2 + \sigma_{xi}^2 + D_e^2\sigma_{\delta e}^2 + \sigma_{xe}^2}\right) \\ \lambda_z &= -P\left(c_x - c_z\frac{D_i^2\sigma_{\delta i}^2 + D_e D_i \sigma_{\delta e}^2}{D_i^2\sigma_{\delta i}^2 + \sigma_{xi}^2 + D_e^2\sigma_{\delta e}^2 + \sigma_{xe}^2}\right) \\ c_x &= \frac{N_e r_e^2 Z_i^2 m_e c \Lambda_C \eta}{\pi\gamma^4\beta^3 A_i m_p \sigma_{\theta e}^2\sigma_{\delta e}}\Phi\left(\frac{\sigma_{\delta e}}{\gamma\sigma_{\theta e}}\frac{\sigma_{xe}}{\sqrt{D_e^2\sigma_{\delta e}^2 + \sigma_{xe}^2}}\right) \\ c_z &= 2\frac{N_e r_e^2 Z_i^2 m_e c \Lambda_C \eta}{\pi\gamma^4\beta^3 A_i m_p \sigma_{\theta e}^2\sigma_{\delta e}}\left[1 - \Phi\left(\frac{\sigma_{\delta e}}{\gamma\sigma_{\theta e}}\frac{\sigma_{xe}}{\sqrt{D_e^2\sigma_{\delta e}^2 + \sigma_{xe}^2}}\right)\right] \\ P &= \frac{\sqrt{D_e^2\sigma_{\delta e}^2 + \sigma_{xe}^2}}{\sigma_{xe}\sqrt{D_i^2\sigma_{\delta i}^2 + \sigma_{xi}^2 + D_e^2\sigma_{\delta e}^2 + \sigma_{xe}^2}\sqrt{\sigma_{xe}^2 + \sigma_{xi}^2}\sqrt{\sigma_{ye}^2 + \sigma_{yi}^2}} \end{aligned}, \quad (16)$$

where duty factor $\eta = L_{CS}/C_{ring}$ and C_{ring} is the storage ring circumference.

Equations (16) give the explicit expressions for redistribution of cooling rates between longitudinal and horizontal direction in presence of electron and ion dispersions in the cooling section. The equations take into account nonuniformity of e-bunch density distribution. Therefore, even with $D_e = 0$ some redistribution is present (non-zero term $D_i^2\sigma_{\delta i}^2$ in the numerator of the redistribution coefficient in the two first equations).

It is important to stress that coefficients c_x and c_z are themselves functions of D_e. Hence, generally speaking, one cannot simply calculate the "undisturbed" cooling rates $(\lambda_{x0}, \lambda_{z0})$ and plug them into simple redistribution equations of the form $\lambda_x = \lambda_{x0} + r\lambda_{z0}$; $\lambda_x = \lambda_{x0} - r\lambda_{z0}$. To the best of our knowledge, this fact was ignored in the previous works dedicated to studies of cooling redistribution.

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SPECIAL CASES

Absence of Redistribution ($D_e = D_i = 0$)

We first consider the simplest case of $D_e = D_i = 0$, which allows us to compare the derived formulas to two well-known asymptotic cases. Zeroing the terms containing dispersion in the friction force equation (7) and using the peak electron density in the beam frame n_e, we get:

$$F_{x,y} = -2\sqrt{2} n_e r_e^2 m_e c^4 Z_i^2 \Lambda_C \frac{v_{xi,yi}}{\sigma_{v\perp e}^2 \sigma_{vze}} \Phi\left(\frac{\sigma_{vze}}{\sigma_{v\perp e}}\right)$$
$$F_z = -4\sqrt{2} n_e r_e^2 m_e c^4 Z_i^2 \Lambda_C \frac{v_{zi}}{\sigma_{v\perp e}^2 \sigma_{vze}}\left[1 - \Phi\left(\frac{\sigma_{vze}}{\sigma_{v\perp e}}\right)\right] . \quad (17)$$

For the case of spherically symmetric velocity distribution ($\sigma_{vze} = \sigma_{v\perp e} \equiv \sigma_{ve}$) $\Phi(1) = 2/3$ and we get:

$$F_{x,y} = F_z = -\frac{4\sqrt{2\pi}}{3} \frac{n_e r_e^2 m_e c^4 Z_i^2 \Lambda_C}{\sigma_{ve}^3} v_{xi,yi,zi} . \quad (18)$$

For the case of $\sigma_{vze} \ll \sigma_{v\perp e}$, which is typical for the coolers utilizing DC electron beam, $\lim_{d\to 0} \Phi(d) = \frac{\pi d}{2}$. Then, from Eq. (17) we get:

$$F_{x,y} = -\pi\sqrt{2\pi} \frac{n_e r_e^2 m_e c^4 Z_i^2 \Lambda_C}{\sigma_{v\perp e}^3} v_{xi,yi}$$
$$F_z = -4\sqrt{2\pi} \frac{n_e r_e^2 m_e c^4 Z_i^2 \Lambda_C}{\sigma_{v\perp e}^2 \sigma_{vze}} v_{zi} . \quad (19)$$

Both equations (18) and (19) coincide with the expressions derived in [1].

Zero Electron Dispersion ($D_e = 0$)

For the case of $D_e = 0$ the redistribution is caused by a nonuniformity of the electron density only (as was discussed above), and Eq. (16) becomes:

$$\lambda_x = -P_0\left(c_{x0} + c_{z0} \frac{D_i^2 \sigma_{\delta i}^2}{D_i^2 \sigma_{\delta i}^2 + \sigma_{xi}^2 + \sigma_{xe}^2}\right)$$
$$\lambda_z = -P_0\left(c_{x0} - c_{z0} \frac{D_i^2 \sigma_{\delta i}^2}{D_i^2 \sigma_{\delta i}^2 + \sigma_{xi}^2 + \sigma_{xe}^2}\right)$$
$$c_{x0} = \frac{N_e r_e^2 Z_i^2 m_e c \Lambda_C \eta}{\pi \gamma^4 \beta^3 A_i m_p \sigma_{\theta_e}^2 \sigma_{\delta e}} \Phi\left(\frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}}\right) . \quad (20)$$
$$c_{z0} = 2\frac{N_e r_e^2 Z_i^2 m_e c \Lambda_C \eta}{\pi \gamma^4 \beta^3 A_i m_p \sigma_{\theta_e}^2 \sigma_{\delta e}}\left[1 - \Phi\left(\frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}}\right)\right]$$
$$P_0 = \frac{1}{\sqrt{D_i^2 \sigma_{\delta i}^2 + \sigma_{xi}^2 + \sigma_{xe}^2}\sqrt{\sigma_{xe}^2 + \sigma_{xi}^2}\sqrt{\sigma_{ye}^2 + \sigma_{yi}^2}}$$

Redistribution formulas similar to Eq. (20) were first derived in [6], although, without specifying the explicit expressions for c_{x0}, c_{z0}.

Uniform Electron Density

The case of a uniform electron density can be obtained from Eq. (16) by assuming $\sigma_{xi}^2 + D_i^2 \sigma_{\delta i}^2 \ll \sigma_{xe}^2 + D_e^2 \sigma_{\delta e}^2$, $\sigma_{yi} \ll \sigma_{ye}$, $\sigma_{zi} \ll \sigma_{ze}$. Then, we get:

$$\lambda_x = \lambda_{x1} + \frac{D_i D_e \sigma_{\delta e}^2}{\sigma_{xe}^2 + D_e^2 \sigma_{\delta e}^2} \lambda_{z1}$$
$$\lambda_z = \lambda_{z1} - \frac{D_i D_e \sigma_{\delta e}^2}{\sigma_{xe}^2 + D_e^2 \sigma_{\delta e}^2} \lambda_{z1}$$
$$\lambda_{x1} = c_1 \Phi\left(\frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}} \frac{\sigma_x}{\sqrt{\sigma_{xe}^2 + D_e^2 \sigma_{\delta e}^2}}\right) . \quad (21)$$
$$\lambda_{z1} = 2c_1\left[1 - \Phi\left(\frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}} \frac{\sigma_x}{\sqrt{\sigma_{xe}^2 + D_e^2 \sigma_{\delta e}^2}}\right)\right]$$
$$c_1 = \frac{N_e r_e^2 Z_i^2 m_e c \Lambda_C \eta}{\pi \gamma^4 \beta^3 A_i m_p \sigma_{xe} \sigma_{ye} \sigma_{ze} \sigma_{\theta_e}^2 \sigma_{\delta e}}$$

The first two formulas in Eq. (21) are well-known (see [1, 7] for example). Yet, the previous works ignored dependence of λ_{x1} and λ_{z1} on D_e, which is important for accurate calculation of the redistributed cooling rates.

CONCLUSION

We derived explicit formulas (Eq. (16)) for redistribution of the cooling rates in non-magnetized electron coolers. The derived expressions take into account both the redistribution due to electron and ion dispersions in the cooling section and a nonuniform transverse density of the electron bunch.

The derived equations show that introduction of electron dispersion changes the cooling rates from their “unperturbed” values and that it is the “new”, “perturbed” rates that get redistributed.

The equations shown in this paper allow for accurate calculation of the redistributed cooling rates.

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