Theoretical Study of Emittance Transfer

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Liouville's theorem

- Suppose a dynamical system of multi-particles interacting each other.
- Define a probability density F in Γ-space $(q_1, q_2, \ldots, q_n; p_1, p_2, \ldots, p_n)$; $Fd\Gamma$ ($d\Gamma = dq_1 dq_2 \cdots dq_n dp_1 dp_2 \cdots dp_n$) is the probability of finding the system in the region $(q_i, q_i + dq_i)$ and $(p_i, p_i + dp_i)$ where $i = 1, 2, \ldots, n$.
- Provided that q_i's and p_i's satisfy the Hamiltonian equations of motion, it is possible to show

$$\frac{\partial F}{\partial t} + \sum_{i=1}^{n} \dot{q}_i \frac{\partial F}{\partial q_i} + \sum_{i=1}^{n} \dot{p}_i \frac{\partial F}{\partial p_i} = 0.$$

$n = 3N$ for a charged-particle beam consisting of N particles.

$$\longrightarrow \quad \frac{DF}{Dt} = 0$$

An important statement of the incompressibility of the flow in phase space !

$\int d\Gamma$ is an invariant.

Emittance

Beam emittance $\equiv$ 6D phase-space (μ-space) volume occupied by a beam.

- *Emittance* is the most important parameter representing the beam quality.
- In standard accelerator theories, we explore the dynamical behavior of a beam in 6D phase space spanned by canonical coordinates $(x, y, z\,;\, p_x, p_y, p_z)$.
- The particle distribution function f in μ-space obeys

$$\frac{Df}{Ds} = \frac{\partial f}{\partial s} + x'\frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + z'\frac{\partial f}{\partial z} + p_x{}'\frac{\partial f}{\partial p_x} + p_y{}'\frac{\partial f}{\partial p_y} + p_z{}'\frac{\partial f}{\partial p_z} = \left(\frac{\partial f}{\partial s}\right)_{col}.$$

The collision term appears in μ-space; thus, strictly speaking, the emittance is not conserved even without dissipation. → $\int d\mu$ is not an invariant *!*

Emittance preservation

In many cases of interest to us, the collision term (in other words, the effects of intra-beam scattering) is negligible because an ordinary beam is very hot and thin.

↓

$$\frac{Df}{Ds} \approx 0$$: Emittance is an approximate invariant in accelerators !

- ✔ In order to improve the beam quality, we have to introduce dissipative interactions (***cooling effect***) into an accelerator.
- ✔ Typical electromagnetic elements (magnets, rf cavities, etc.) only provide conservative forces that never affect the 6D volume of a beam.
- ✔ Beam cooling is very difficult in practice.

→ Only few cooling techniques have been developed so far.

Why cooling?

- The volume occupied by a beam in phase space is the measure of ***beam quality***.

  → A smaller volume means a higher quality.

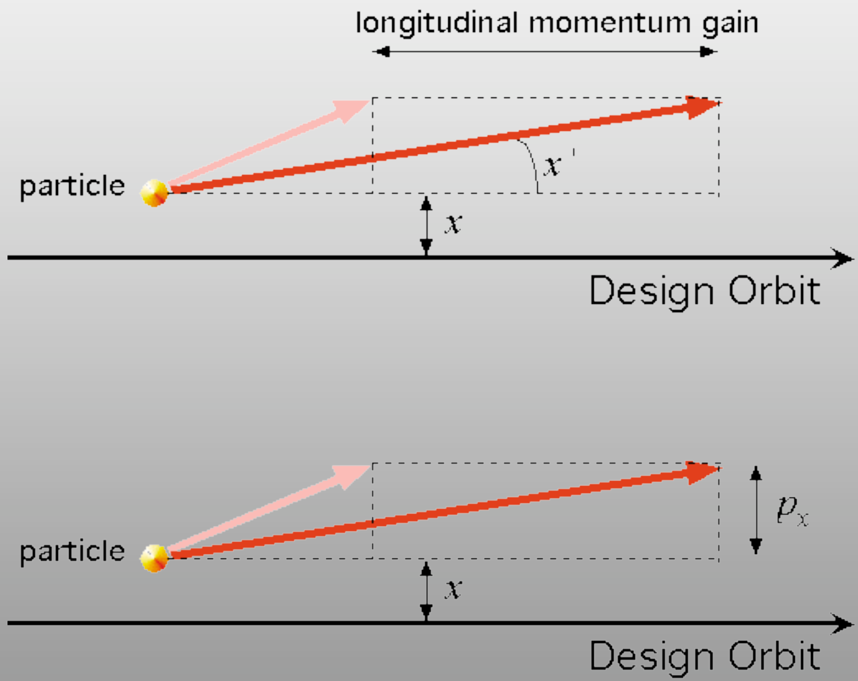
- A particle accelerator is, in general, a conservative dynamical system.

  → **The beam volume in 6D phase space is an approximate invariant.**

We have to introduce a dissipative force into an accelerator, in order to improve the beam quality for users.

Cooling and damping

- Transverse projected emittances conventionally defined in x-x' and y-y' phase planes damp when the beam is accelerated (adiabatic damping).
- In one word, this is because a pair of variables (x, x') or (y, y') are not canonical conjugate where the prime stands for s-derivative, i.e. $' \equiv d/ds$.
- Since a beam is accelerated in general by a conservative force (such as RF electric fields), the 6D emittance is certainly unchanged.



Do not confuse adiabatic damping with cooling !

Non-dissipative manipulation of μ-space

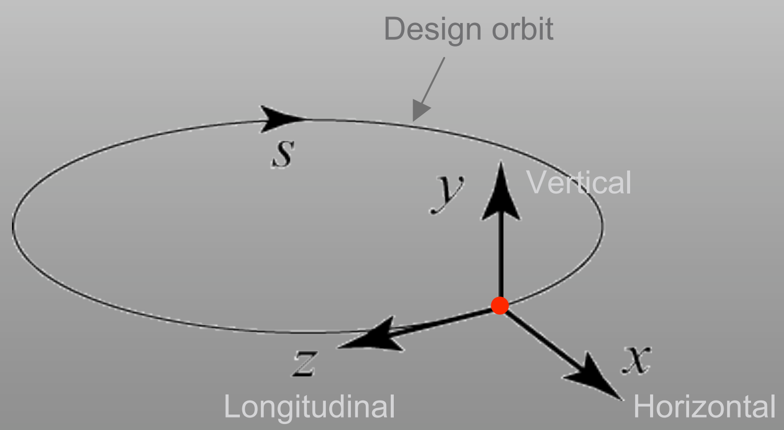
Concept of emittance manipulation

"Cooling" is difficult because the 6D volume
of a beam is invariant in Hamiltonian systems.

↕

It is much easier to manipulate 2D subspaces (emittance projections).

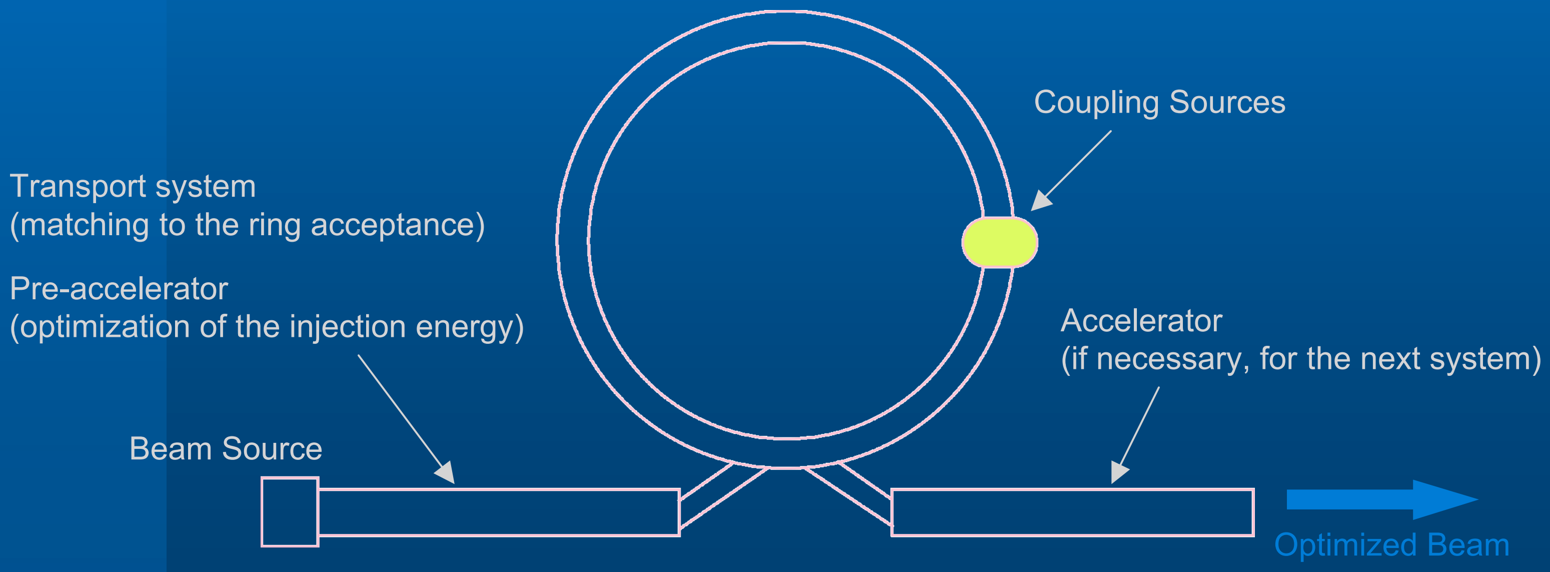
- We often consider projections of the 6D emittance onto 2D phase planes.
- *Projected emittances* of a beam are defined in x-p_x, y-p_y, and z-p_z phase planes.
- In general, projected emittances are *not* invariant; it is easy to change them by introducing coupling potentials that correlate the degrees of freedom.



Coupling storage ring

A COMPACT STORAGE RING FOR THE OPTIMIZATION OF CHARGED-PARTICLE BEAMS IN 6D PHASE SPACE

- 2D subspaces are connected through various coupling potentials artificially controllable.
- Not only a full emittance exchange but also a partial emittance transfer are feasible.
- It is possible to develop various correlations between the three degrees of freedom.



Model

For the sake of simplicity, ignore the following effects:

- Interparticle interactions through the Coulomb fields
- Details of the lattice design (smooth approximation)
- Radiation, wakefields, etc.

The Hamiltonian of the dynamical system of interest to us can then be written as

$$H = \frac{1}{2}\left[p_x^2 + \left(\frac{\nu_x}{R}\right)^2 x^2\right] + \frac{1}{2}\left[p_y^2 + \left(\frac{\nu_y}{R}\right)^2 y^2\right] + \frac{1}{2}\left[p_z^2 + \left(\frac{\nu_z}{R}\right)^2 z^2\right] + \phi_c(x, y, z; s).$$

(ν_x, ν_y, ν_z) : tunes
R : average radius of the coupling ring
ϕ_c : artificial coupling potential

Example:

$$\phi_c = g_1 x^m z^n \delta_p(s - s_1) + g_2 y^m z^n \delta_p(s - s_2)$$

In general, the coupling constants g_1 and g_2 are s-dependent; it is possible to change gradually or switch on and off individual coupling terms.

Resonant emittance transfer

- The coupling between the degrees of freedom can be strengthened by increasing the coupling constants.
- The enhancement of coupling is, however, almost useless for the present purposes.
- In order to achieve an efficient emittance transfer, the coupling ring must operate near resonances:

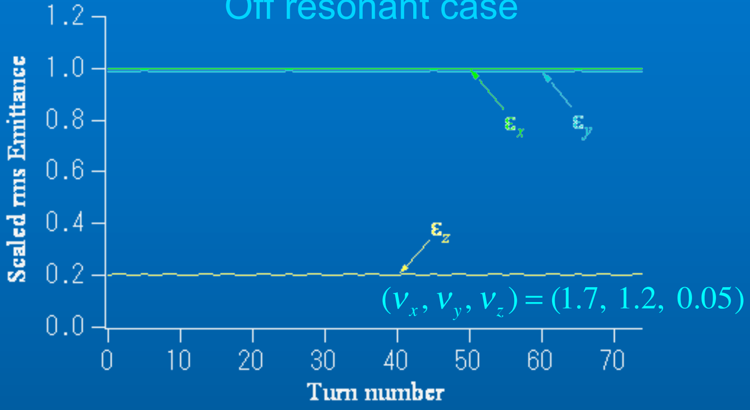
$$m\nu_x - n\nu_z \approx \text{integer},$$

$$m\nu_y - n\nu_z \approx \text{integer},$$

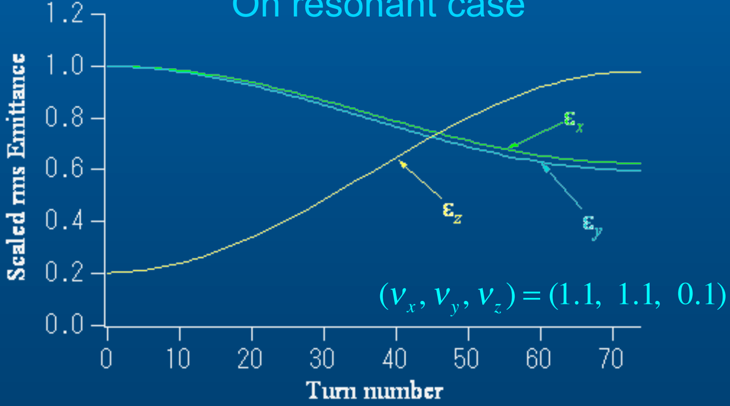
for the case where

$$\phi_c = g_1 x^m z^n \delta_p(s - s_1) + g_2 y^m z^n \delta_p(s - s_2).$$

Off resonant case



On resonant case



In both cases, $R = 1$ and $g_1 = g_2 = 0.01$.

Invariants

- 2D (g_1 or $g_2 = 0$)

$$I_{2D} \equiv \frac{\varepsilon_{x(y)}}{m} + \frac{\varepsilon_z}{n} = \text{const.}$$

- 3D (full coupling)

$$I_{3D} \equiv \frac{\varepsilon_x + \varepsilon_y}{m} + \frac{\varepsilon_z}{n} = \text{const.}$$

We have again assumed:

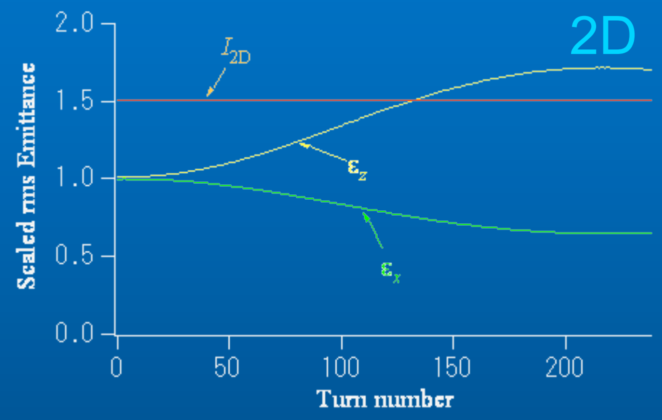
$$\phi_c = g_1 x^m z^n \delta_p(s - s_1) + g_2 y^m z^n \delta_p(s - s_2).$$

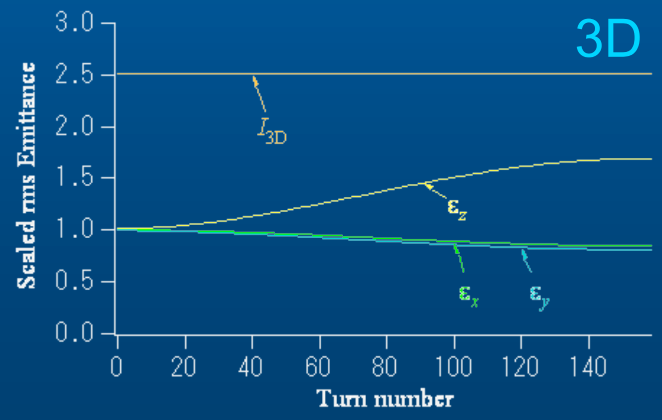
Different forms of the invariant can be found for different coupling potentials.

Scaling law : $$g\left(\frac{R}{\nu_\perp}\right)^{\frac{m}{2}}\left(\frac{R}{\nu_z}\right)^{\frac{n}{2}} = \text{const.}$$

$$(g \equiv g_1 = g_2,\ \nu_\perp \equiv \nu_x = \nu_x)$$

Third-order emittance transfer ($m = 1, n = 2$)





Equilibrium emittance ratio

In a 2D case where $g_2 = 0$, we can find another invariant by averaging the single-particle motion near resonance, i.e. $\Delta \equiv (m\nu_x - n\nu_z - \ell)/R \approx 0$:

$$\bar{H} = \begin{cases} \Delta J \pm h_{mn} J^{n/2}(1-J)^{m/2} \cos\psi & \text{for} \quad m+n = \text{even}, \\ \Delta J \pm h_{mn} J^{n/2}(1-J)^{m/2} \sin\psi & \text{for} \quad m+n = odd, \end{cases}$$

where h_{mn} is a constant parameter, $J = \varepsilon_z / nI_{2\mathrm{D}}$ and ψ is the relative phase variable conjugate to J. $\bar{H}$ can be considered as the Hamiltonian of the averaged system. The equilibrium emittances are then given by

$$\varepsilon_x \approx \frac{m^2}{m+n} I_{2D}, \quad \varepsilon_z \approx \frac{n^2}{m+n} I_{2D}.$$

Assuming that an incident beam eventually come close to this equilibrium state,

$$\frac{\varepsilon_x}{\varepsilon_x(0)} \to \frac{1 + \frac{m}{n}\frac{\varepsilon_z(0)}{\varepsilon_x(0)}}{1 + \frac{n}{m}}, \qquad \frac{\varepsilon_z}{\varepsilon_z(0)} \to \frac{1 + \frac{n}{m}\frac{\varepsilon_x(0)}{\varepsilon_z(0)}}{1 + \frac{m}{n}}.$$

Emittance transfer and correlation

☆ Example (linear case)

Second-moment matrix: $\mathbf{S} = \begin{pmatrix} \mathbf{S}_{xx} & \mathbf{S}_{xy} \\ \mathbf{S}_{yx} & \mathbf{S}_{yy} \end{pmatrix}$

The off-diagonal sub-matrices $\mathbf{S}_{xy}$ and $\mathbf{S}_{yx}$ describe correlation.

where $\mathbf{S}_{ij}$ $(i, j = x, y, z)$ are 2x2 sub-matrices such as $\mathbf{S}_{xx} = \begin{pmatrix} \langle x^2 \rangle & \langle x p_x \rangle \\ \langle p_x x \rangle & \langle p_x^2 \rangle \end{pmatrix}$.

- ✔ *Projected emittances are defined as* $\varepsilon_x^2 = \det \mathbf{S}_{xx}$ *and* $\varepsilon_y^2 = \det \mathbf{S}_{yy}$.
- ✔ *The emittance (the 4D volume) is invariant and given by* $\varepsilon_{4\mathrm{D}}^2 = \det \mathbf{S}$.
- ✔ *With a transfer matrix* $\mathbf{M}$, *the second moments are transformed to* $\mathbf{M}\mathbf{S}\mathbf{M}^T$.

⬇

Even if we start with an ordinary “uncorrelated” beam, the elements of the off-diagonal matrices become non-zero during an emittance transfer process where $\varepsilon_{4\mathrm{D}}$ and $\varepsilon_x + \varepsilon_y$ are both conserved. (At each moment of a full emittance exchange, the correlation disappears.)

Linear correlation in phase space

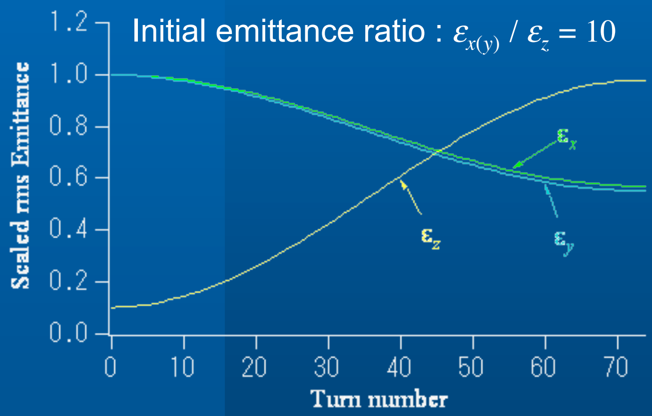
Example

Linear coupling ($m = n = 1$)

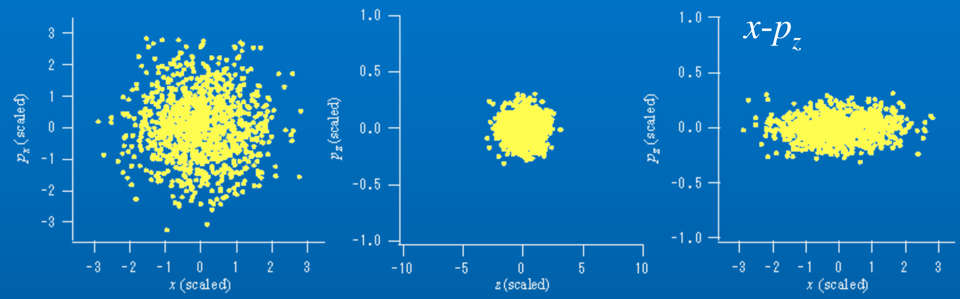
$(\nu_x, \nu_y, \nu_z) = (1.1,\ 1.1,\ 0.1)$

$R = 1$

$g_1 = g_2 = 0.01$

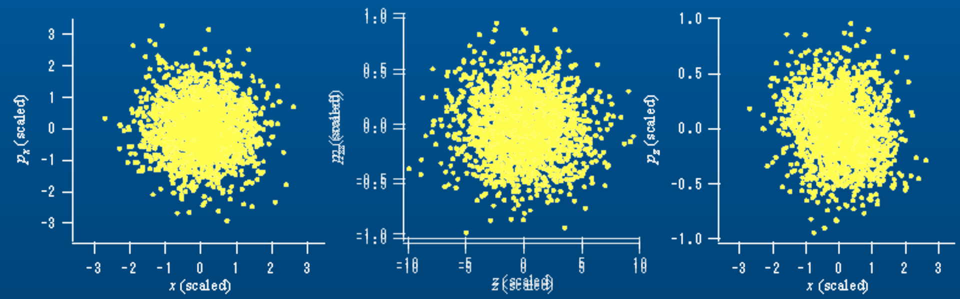


INITIAL DISTRIBUTION



DISTRIBUTION AFTER A FULL EMITTANCE EXCHANGE

DISTRIBUTION AT 40TH TURN



Nonlinear correlation in phase space

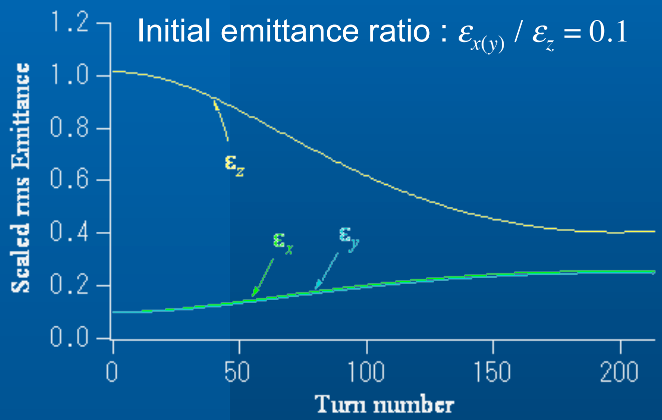
Example

Nonlinear coupling ($m = 1, n = 2$)

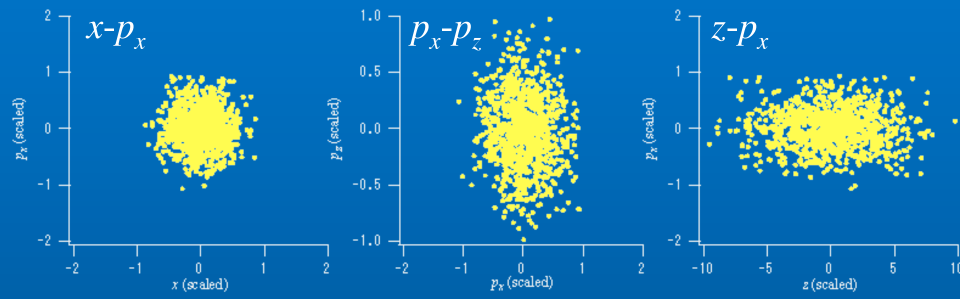
$(\nu_x, \nu_y, \nu_z) = (1.2,\ 1.2,\ 0.1)$

$R = 1$

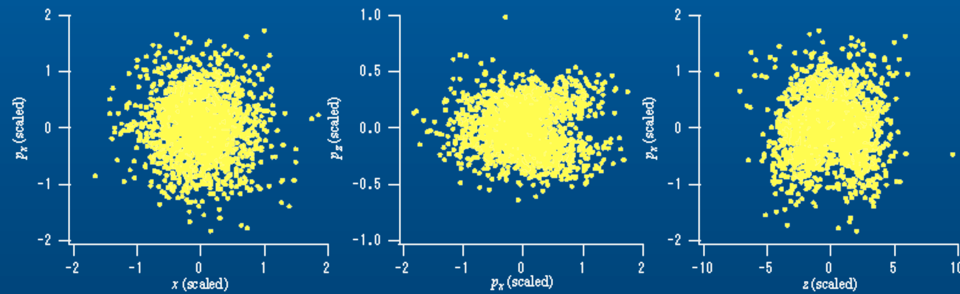
$g_1 = g_2 = 1$



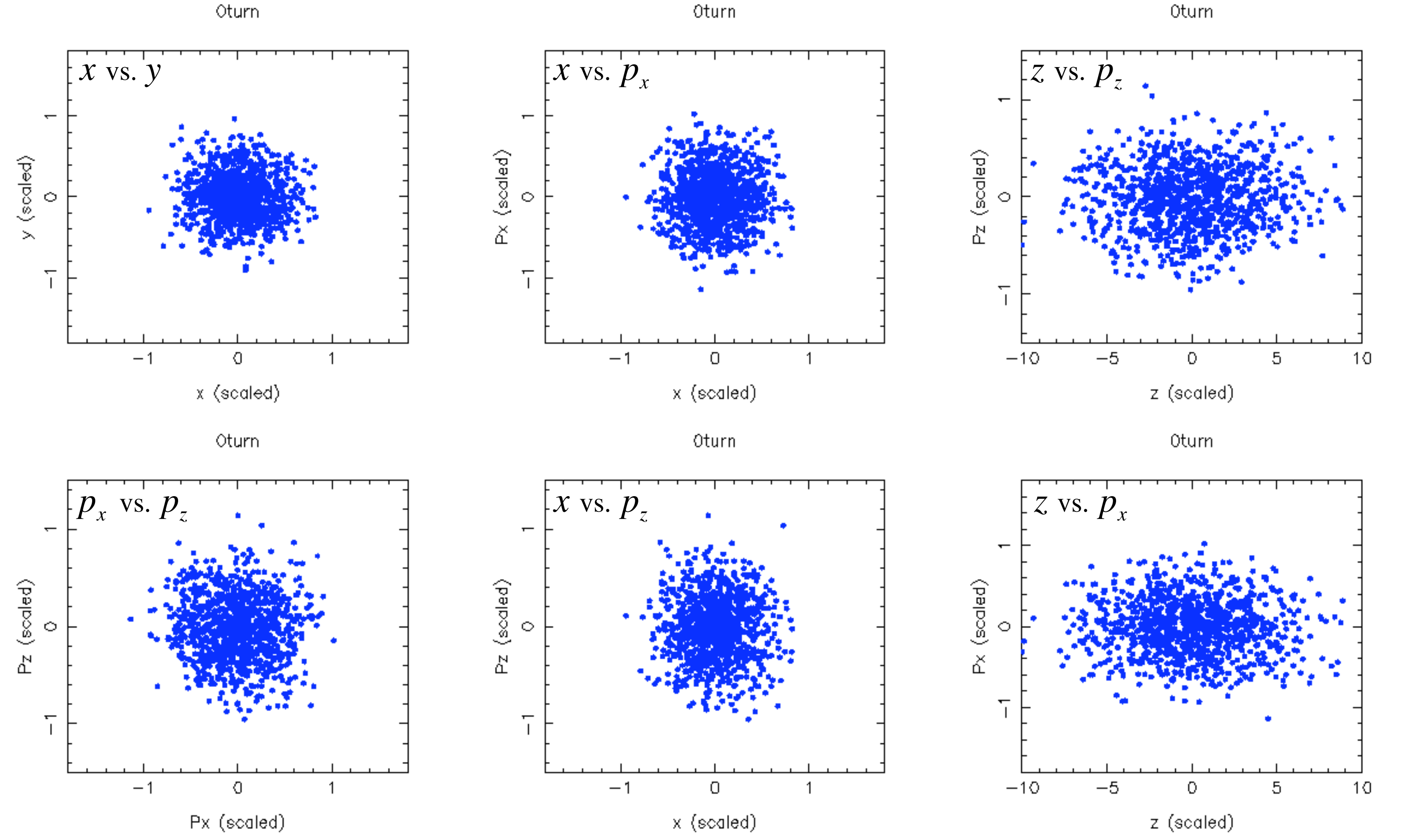
INITIAL DISTRIBUTION



DISTRIBUTION AT 226TH TURN



Phase-space evolution



Coupling sources

The electromagnetic fields of typical coupling sources can be derived from vector potentials of the form $\mathbf{A} = (0, \quad 0, \quad A_z)$.

- Transverse - transverse

  It is straightforward to develop linear and nonlinear couplings between the two transverse directions; we can employ various magnets such as a solenoid (linear), a skew quadrupole (linear), a sextupole (third-order), an octupole (fourth-order), etc.

  For instance, $A_z = g_{skew} xy\delta_p(s)$ for a skew quadrupole magnet.

- Transverse - longitudinal

  In order to couple the transverse motion with the longitudinal motion, we employ radio-frequency devices or momentum dispersion.

  Linear coupling rf cavity (rectangular) : $A_z \approx g_{rf} x \sin(\omega t)\delta_p(s)$

  Nonlinear coupling rf cavity (cylindrical) : $A_z \approx g_{rf} J_m\left(\zeta_{mn}\frac{r}{r_0}\right)\cos(m\theta)\sin(\omega t)\delta_p(s)$

  (TM_{mn0}-mode operation)

Hamiltonian

- Coupling potential

$$A_z = \frac{p_0 \Gamma_q}{qR} xy\delta_p(s-s_q) + \frac{V_1}{\omega}\left(\frac{\pi x}{a}\right)\sin(\omega t+\phi_1)\delta_p(s-s_1) + \frac{V_2}{\omega}\left[1-\left(\frac{\zeta_{01}}{2}\frac{r}{r_0}\right)^2\right]\sin(\omega t+\phi_2)\delta_p(s-s_2)$$

(Skew quadrupole) (Linear coupling cavity) (Third-order coupling cavity)

- Hamiltonian in betatron space

$$\begin{aligned}\hat{H} =& \frac{1}{2}\left(\frac{1}{\gamma_0^2}-\frac{D_x}{\rho}\right)\left(\frac{\omega\Delta\hat{E}}{\beta_0 c}\right)^2 + \frac{\hat{p}_x^2+\hat{p}_y^2}{2} + \frac{1}{2}(K_x\hat{x}^2+K_y\hat{y}^2) \\ &+ \frac{qV_b}{2\omega p_0}\left[\hat{\psi}-\frac{\omega}{\beta_0 c}(D_x\hat{p}_x - D_x'\hat{x})\right]^2\delta_p(s-s_b) + \frac{\Gamma_q}{R}\left(\hat{x}+\frac{\omega D_x}{\beta_0 c}\Delta\hat{E}\right)\hat{y}\delta_p(s-s_q) \\ &- \frac{\pi q V_1}{a\omega p_0}\left(\hat{x}+\frac{\omega D_x}{\beta_0 c}\Delta\hat{E}\right)\left[\hat{\psi}-\frac{\omega}{\beta_0 c}(D_x\hat{p}_x - D_x'\hat{x})\right]\delta_p(s-s_1) \\ &- \frac{qV_2}{\omega p_0}\left[1-\left(\frac{\zeta_{01}}{2r_0}\right)^2\left(\hat{x}+\frac{\omega D_x}{\beta_0 c}\Delta\hat{E}\right)^2-\left(\frac{\zeta_{01}}{2r_0}\right)^2\hat{y}^2\right]\left[\hat{\psi}-\frac{\omega}{\beta_0 c}(D_x\hat{p}_x - D_x'\hat{x})\right]\delta_p(s-s_2).\end{aligned}$$

Canonical variables : $(\hat{x}, \hat{y}, \hat{\psi}; \hat{p}_x, \hat{p}_y, -\Delta\hat{E})$

Application to FEL

Necessary conditions

Resonance Condition for FEL : $\lambda = \left(1 - \frac{v_z}{c}\right)\lambda_w \approx \frac{1+K^2}{2\gamma^2}\lambda_w$

(K is the normalized strength of the undulator.)

- The transverse normalized emittance ε_n of the electron beam must be less than $\lambda / 4\pi$.
- The energy spread of the electron beam must be less than the FEL parameter ρ.
- The current of the electron beam should be high so as to make the FEL parameter large and the saturation distance short.

⬇

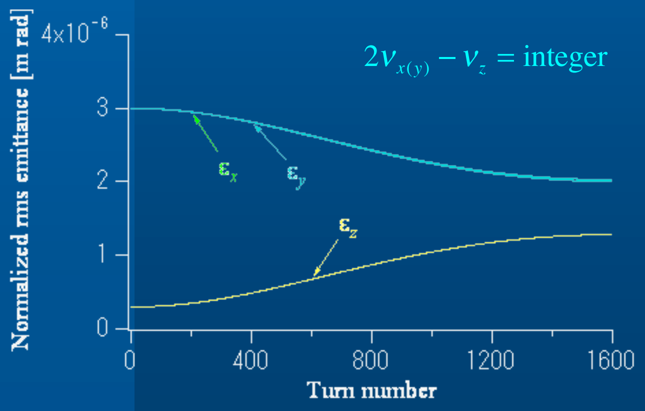
These conditions suggest that an optimization of the three projected emittances may improve the performance of an FEL system.

Emittance optimization

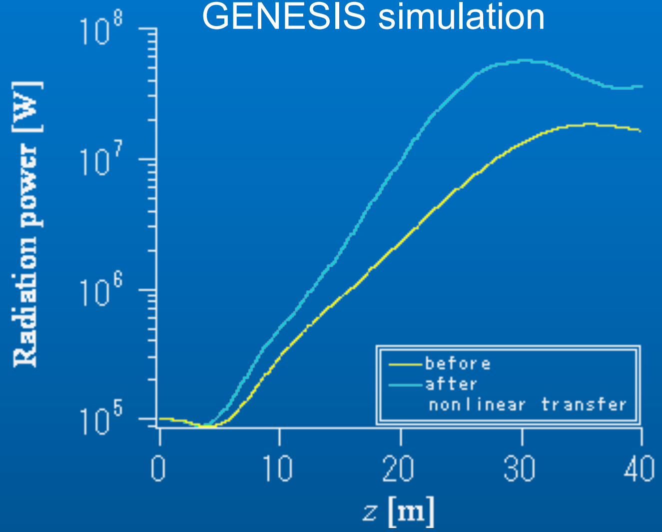
- The recent progress of accelerator technologies has made it feasible to produce a high-quality electron beam with a very low longitudinal emittance.
- The FEL gain is then limited mainly by the transverse beam quality.

Reduce ε_x and ε_y at the sacrifice of ε_z !

Example: third-order emittance transfer



GENESIS simulation



Radiation wavelength λ [nm]	6
Beam energy [GeV]	1
Peak current [A]	250
Undulator period λ_w [cm]	2
Undulator parameter K	1.14

Beam conditioning

Ideal resonant wavelength : $\lambda = \left(1 - \frac{v_z}{c}\right)\lambda_w \approx \frac{1+K^2}{2\gamma^2}\lambda_w$

- The FEL resonance condition for an electron with a finite transverse amplitude can be given by

$$\lambda \approx \frac{1}{2}\left(\frac{1+K^2}{\gamma^2} + \frac{2J_x}{\beta_x} + \frac{2J_y}{\beta_y}\right)\lambda_w .$$

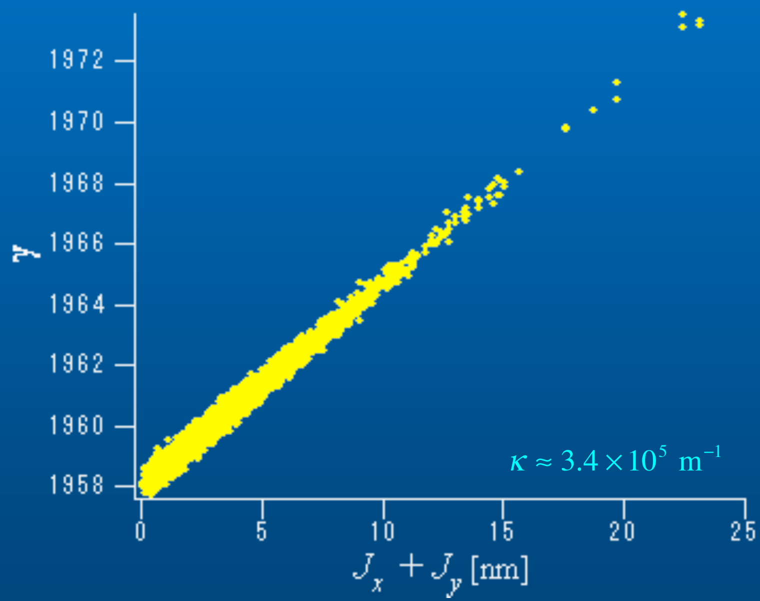
where $J_{x(y)}$ are the transverse actions.

- The energy (γ) of each individual electron is different from the ideal design value (γ_0).

- Substituting $\Delta\gamma = \gamma - \gamma_0$ into the above equation, we find that the number of resonant electrons increases considerably when the beam has the following correlation:

$$\frac{\Delta\gamma}{\gamma} = \kappa_x J_x + \kappa_x J_y ,$$

where $\kappa_{x(y)}$ are the conditioning parameters; when $\kappa_x = \kappa_y$, we have $\kappa_x = (\lambda_w / \lambda) / 2\beta_x$.



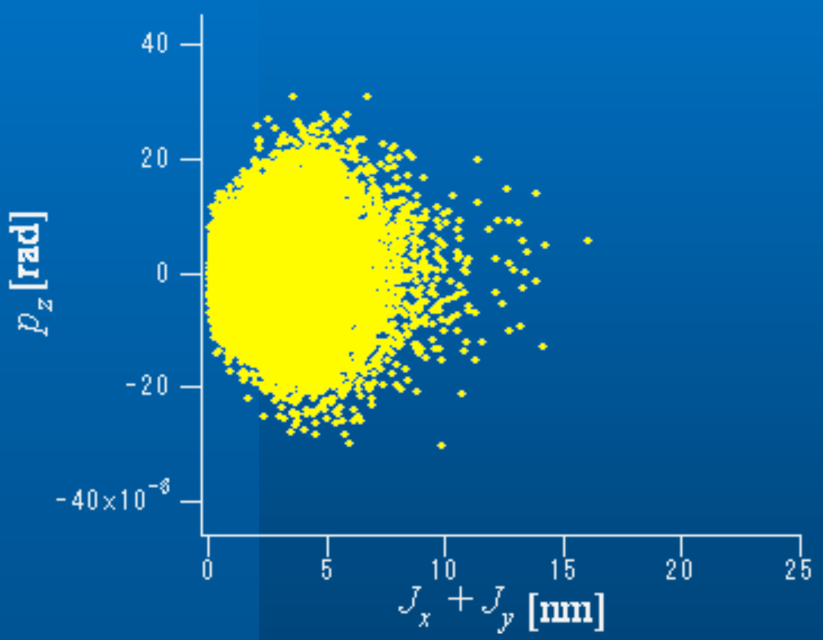
Radiation wavelength $\lambda = 6$ nm

$\lambda_w = 2$ cm ; $K = 1.14$ nm ; Beam energy=1 GeV

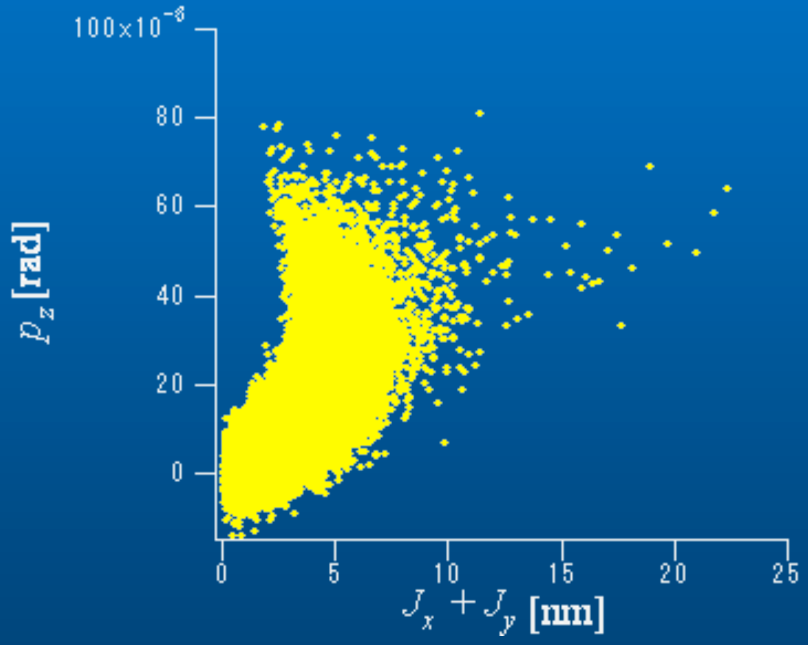
Transverse normalized emittance $\varepsilon_x = \varepsilon_y = 3\ \mu$m

A possible conditioning in a coupling ring

Third-order emittance transfer $(m = 2, n = 1)$

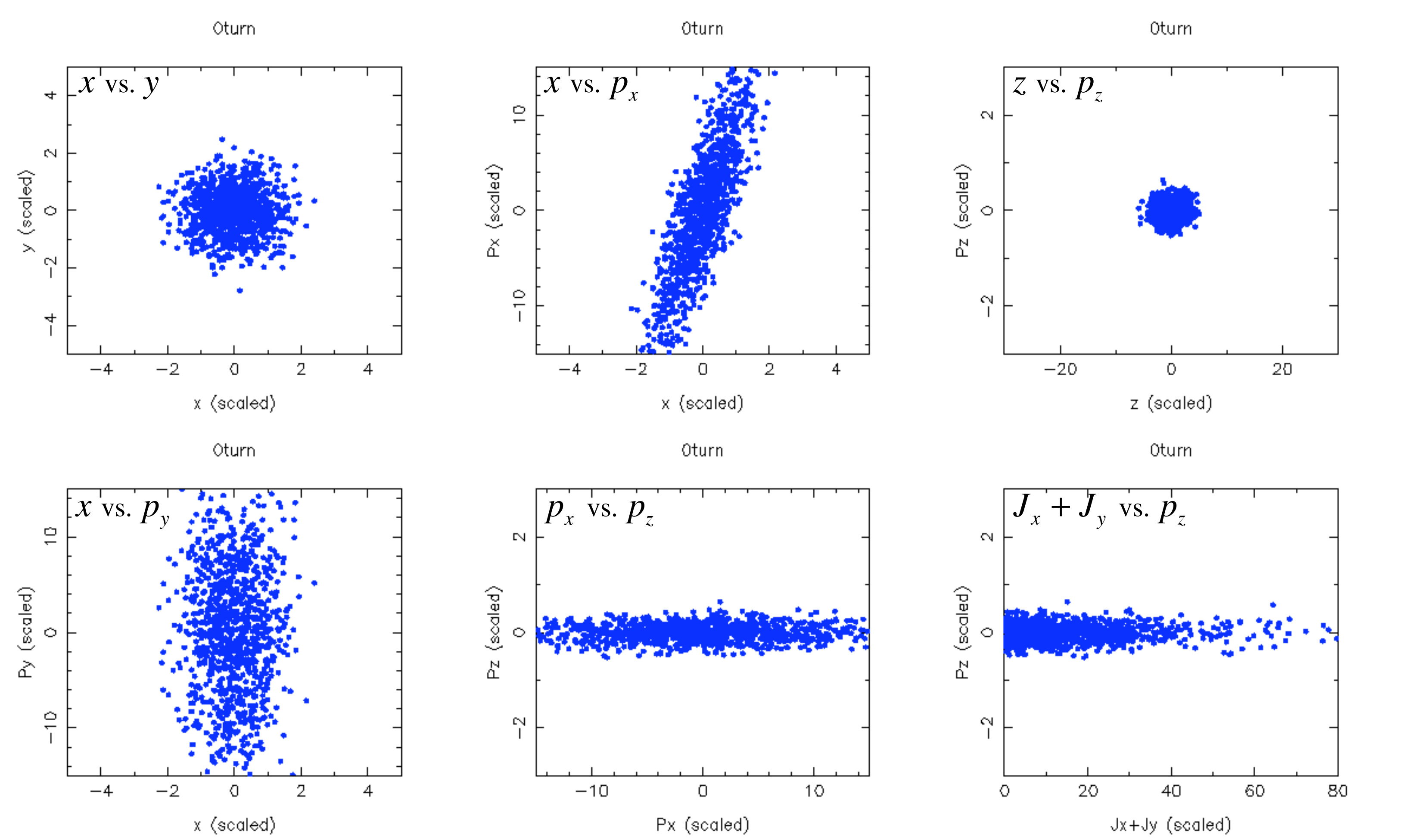


Initially matched

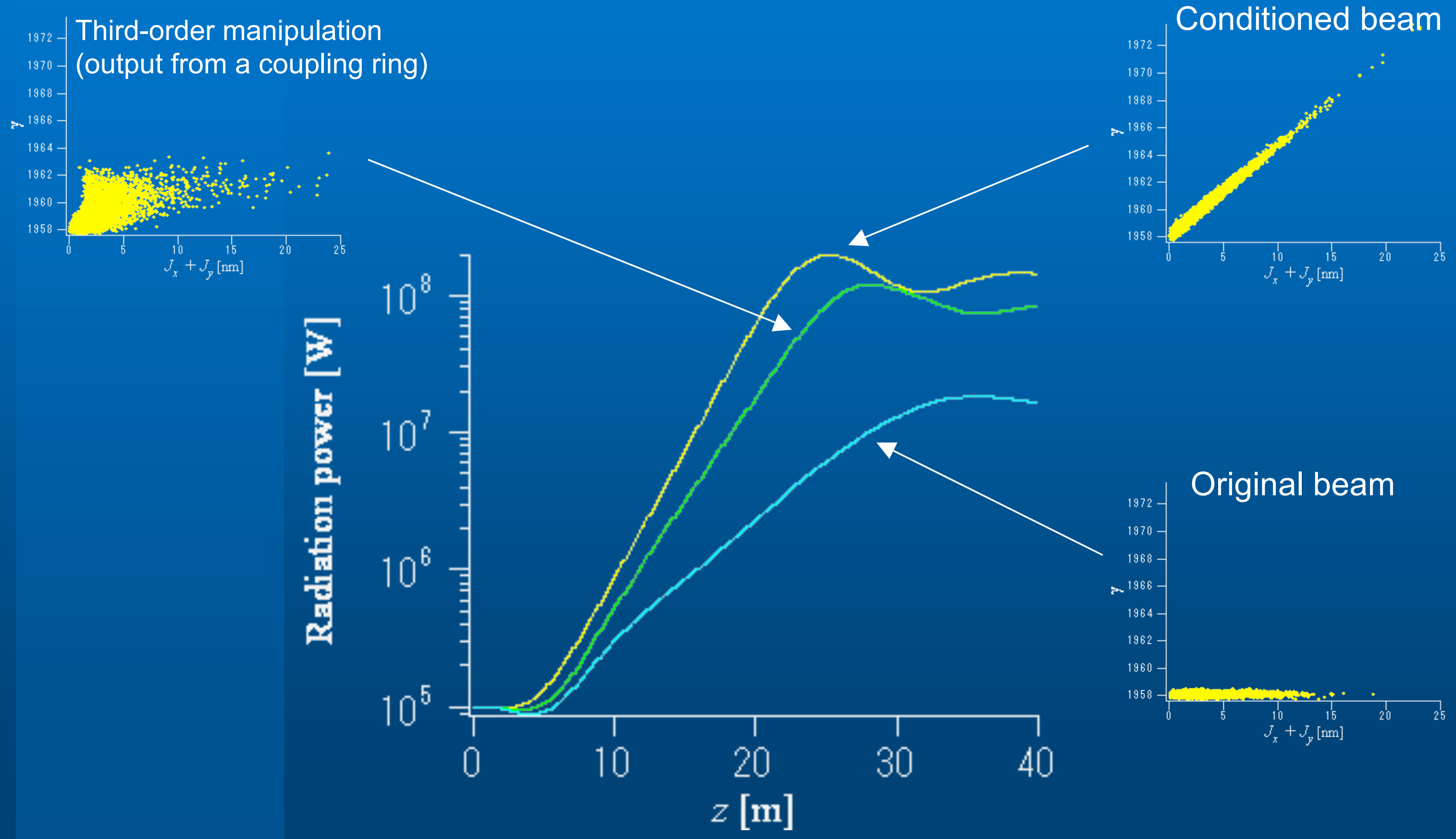


Mismatched

Phase-space evolution (mismatched)



GENESIS result



Summary

- Non-dissipative manipulation of charged-particle beams should be useful in practice.
- An efficient emittance transfer is achievable by resonantly coupling the degrees of freedom.
- A coupling storage ring enables one to optimize the ratios of three projected emittances for specific purposes.
- Various linear and nonlinear correlations in phase space can naturally be developed during an emittance transfer process.

The present method thus offers a possibility of controlling charged-particle beams in 6D phase space; an optimization of not only emittance ratios but also detailed structures of particle distributions in subspaces may be feasible.