NONLINEAR TRANSPORT OF ACCELERATOR BEAM PHASE SPACE

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ABSTRACT

Accelerator beam is a nonlinear dynamic system, and the phase space transport of it means the variation of the phase space shape from the initial state $X_0 = X(t_0)$ to the final state $X_t = X(t)$. Mathematically, this variation is the transform from the initial boundary equation of phase space $F(X_0) = 0$ to the final equation $F(X_0(X_t, t)) = f(X_t, t) = 0$. This transform is nonlinear. The any order analytical solution for the motion differential equation of a general nonlinear dynamic system has been derived out[1], namely, the function relation $X(t) = X_t(X_0, t)$ was given. Based on the result, the inverse transform of the any order solution of the motion differential equation was completed in this paper, and the function relation $X(t_0) = X_0(X_t, t)$ was obtained, thus realizing theoretically the nonlinear transport for the phase space of nonlinear dynamic system.

1. INTRODUCTION

With the development of modern science, more and more science areas have broken through the linear category and involved the studies of nonlinear category. The phase space transport of nonlinear dynamic systems is a general nonlinear problem. At the initial state $X_0 = X(t_0)$, if the boundary equation of the phase space is

$$F(X_0) = 0 \ , \tag{1}$$

then, at the final state $X_t = X(t)$, the boundary equation of the phase space becomes

$$F[X_0(X_t, t)] = f(X_t, t) = 0 \ , \tag{2}$$

the process between the initial phase space Eq.1 and the finial one Eq.2 is the very phase space transport of the nonlinear dynamic system considered. Mathematically, this physical process is the nonlinear transformation from the given initial Eq.1 to the finial Eq.2. For a general nonlinear dynamic system, reference 1) gives the phase point $X(t) = [x_t^1, x_t^2, \cdots, x_t^N]^T = X_t(X_0, t)$ at the time t as the funtion of the initial phase point $X_0 = [x_0^1, x_0^2, \cdots, x_0^N]^T$ and the time t. So, our task is to find out its inverse transformation, i.e. the initial phase point $X(t_0) = X_0(X_t, t)$ as the funtion of the phase point X_t and the time t. Then, substitution of the inverse transform $X(t_0) = X_0(X_t, t)$ into the initial Eq.1 yields directly the final state Eq.2.

2. DERIVING THE INVERSE TRANSFORMATION OF $X(t) = X_t(X_0, t)$

In N-dimension phase space, the motion differential equation of a general nonlinear dynamic system is

$$\begin{aligned} \dot{x}_t^i = \ & \alpha^i_{j_1}(t) x_t^{j_1} + \alpha^i_{j_1 j_2}(t) x_t^{j_1} x_t^{j_2} \\ & + \alpha^i_{j_1 j_2 j_3}(t) x_t^{j_1} x_t^{j_2} x_t^{j_3} \\ & + \cdots \\ & + \alpha^i_{j_1 j_2 \ldots j_n}(t) x_t^{j_1} x_t^{j_2} \ldots x_t^{j_n} \\ & + \cdots \ , \end{aligned} \tag{3}$$

where, i = 1, 2, ..., N, and the any order analytical solution[1)] of the above equation with initial phase point X_0^i is

$$\begin{aligned} x_t^i = \ & T^i_{j_1}(t) x_0^{j_1} + T^i_{j_1 j_2}(t) x_0^{j_1} x_0^{j_2} \\ & + T^i_{j_1 j_2 j_3}(t) x_0^{j_1} x_0^{j_2} x_0^{j_3} \\ & + \cdots \\ & + T^i_{j_1 j_2 \ldots j n}(t) x_0^{j_1} x_0^{j_2} \ldots x_0^{j_n} \\ & + \cdots \ . \end{aligned} \tag{4}$$

Express the inverse transform of Eq.4 in series:

$$\begin{aligned} x_0^j = \; & S_{i_1}^j(t)x_t^{i_1} + S_{i_1 i_2}^j(t)x_t^{i_1}x_t^{i_2} \\ & + S_{i_1 i_2 i_3}^j(t)x_t^{i_1}x_t^{i_2}x_t^{i_3} \\ & + \cdots \\ & + S_{i_1 i_2 \ldots i_n}^j(t)x_t^{i_1}x_t^{i_2}\ldots x_t^{i_n} \\ & + \cdots \ , \end{aligned} \qquad (5)$$

or:

$$\left\{ \begin{aligned} x_0^{j_1} = \; & S_{i_1}^{j_1}(t)x_t^{i_1} + S_{i_1 i_2}^{j_1}(t)x_t^{i_1}x_t^{i_2} \\ & + S_{i_1 i_2 i_3}^{j_1}(t)x_t^{i_1}x_t^{i_2}x_t^{i_3} \\ & + \cdots \\ & + S_{i_1 i_2 \ldots i_n}^{j_1}(t)x_t^{i_1}x_t^{i_2}\ldots x_t^{i_n} \\ & + \cdots \\ x_0^{j_2} = \; & S_{i_1}^{j_2}(t)x_t^{i_1} + S_{i_1 i_2}^{j_2}(t)x_t^{i_1}x_t^{i_2} \\ & + S_{i_1 i_2 i_3}^{j_2}(t)x_t^{i_1}x_t^{i_2}x_t^{i_3} \\ & + \cdots \\ & + S_{i_1 i_2 \ldots i_n}^{j_2}(t)x_t^{i_1}x_t^{i_2}\ldots x_t^{i_n} \\ & + \cdots \\ x_0^{j_3} = \; & S_{i_1}^{j_3}(t)x_t^{i_1} + S_{i_1 i_2}^{j_3}(t)x_t^{i_1}x_t^{i_2} \\ & + S_{i_1 i_2 i_3}^{j_3}(t)x_t^{i_1}x_t^{i_2}x_t^{i_3} + \cdots \\ & + S_{i_1 i_2 \ldots i_n}^{j_3}(t)x_t^{i_1}x_t^{i_2}\ldots x_t^{i_n} \\ & + \cdots \\ \ldots\ldots & \\ x_0^{j_n} = \; & S_{i_1}^{j_n}(t)x_t^{i_1} + S_{i_1 i_2}^{j_n}(t)x_t^{i_1}x_t^{i_2} \\ & + S_{i_1 i_2 i_3}^{j_n}(t)x_t^{i_1}x_t^{i_2}x_t^{i_3} + \cdots \\ & + S_{i_1 i_2 \ldots i_n}^{j_n}(t)x_t^{i_1}x_t^{i_2}\ldots x_t^{i_n} \\ & + \cdots \\ \ldots\ldots & \ . \end{aligned} \right. \qquad (6)$$

giving the fundamental terms:

$$\begin{aligned} x_0^{j_1}x_0^{j_2} & \\ = \; & S_{i_1}^{j_1}(t)S_{i_2}^{j_2}(t)x_t^{i_1}x_t^{i_2} \\ & + \{S_{i_1}^{j_1}(t)S_{i_2 i_3}^{j_2}(t) \\ & + S_{i_1 i_2}^{j_1}(t)S_{i_3}^{j_2}(t)\}x_t^{i_1}x_t^{i_2}x_t^{i_3} \\ & + \cdots \\ & + \{S_{i_1}^{j_1}(t)S_{i_2 i_3 \ldots i_n}^{j_2}(t) \\ & + S_{i_1 i_2}^{j_1}(t)S_{i_3 i_4 \ldots i_n}^{j_2}(t) \\ & + \cdots \\ & + S_{i_1 i_2 \ldots i_{n-2}}^{j_1}(t)S_{i_{n-1} i_n}^{j_2}(t) \\ & + S_{i_1 i_2 \ldots i_{n-1}}^{j_1}(t)S_{i_n}^{j_2}(t)\}x_t^{i_1}x_t^{i_2}\ldots x_t^{i_n} \\ & + \cdots \\ = \; & G_{i_1 i_2}^{j_1 j_2}(t)x_t^{i_1}x_t^{i_2} \\ & + G_{i_1 i_2 i_3}^{j_1 j_2}(t)x_t^{i_1}x_t^{i_2}x_t^{i_3} \\ & + \cdots \\ & + G_{i_1 i_2 \ldots i_n}^{j_1 j_2}(t)x_t^{i_1}x_t^{i_2}\ldots x_t^{i_n} \\ & + \cdots \end{aligned} \qquad (7-1)$$

$$\begin{aligned} x_0^{j_1}x_0^{j_2}x_0^{j_3} = \; & (x_0^{j_1}x_0^{j_2})x_0^{j_3} \\ = \; & \{G_{i_1 i_2}^{j_1 j_2}(t)x_t^{i_1}x_t^{i_2} + G_{i_1 i_2 i_3}^{j_1 j_2}(t)x_t^{i_1}x_t^{i_2}x_t^{i_3} \\ & + \cdots + G_{i_1 i_2 \ldots i_n}^{j_1 j_2}(t)x_t^{i_1}x_t^{i_2}\ldots x_t^{i_n} \\ & + \cdots\}\{S_{i_1}^{j_3}(t)x_t^{i_1} + S_{i_1 i_2}^{j_3}(t)x_t^{i_1}x_t^{i_2} \\ & + S_{i_1 i_2 i_3}^{j_3}(t)x_t^{i_1}x_t^{i_2}x_t^{i_3} + \cdots \\ & + S_{i_1 i_2 \ldots i_n}^{j_3}(t)x_t^{i_1}x_t^{i_2}\ldots x_t^{i_n} + \cdots\} \\ = \; & G_{i_1 i_2}^{j_1 j_2}(t)S_{i_3}^{j_3}(t)x_t^{i_1}x_t^{i_2}x_t^{i_3} + \cdots \\ & + \{G_{i_1 i_2 \cdots i_{n-1}}^{j_1 j_2}(t)S_{i_n}^{j_3}(t) \\ & + G_{i_1 i_2 \ldots i_{n-2}}^{j_1 j_2}(t)S_{i_{n-1} i_n}^{j_3}(t) + \ldots \\ & + G_{i_1 i_2}^{j_1 j_2}(t)S_{i_3 i_4 \ldots i_n}^{j_3}(t)\}x_t^{i_1}x_t^{i_2}\ldots x_t^{i_n} \\ & + \cdots \\ = \; & G_{i_1 i_2 i_3}^{j_1 j_2 j_3}(t)x_t^{i_1}x_t^{i_2}x_t^{i_3} + \cdots \\ & + G_{i_1 i_2 \ldots i_n}^{j_1 j_2 j_3}(t)x_t^{i_1}x_t^{i_2}\ldots x_t^{i_n} \\ & + \cdots \end{aligned} \qquad (7-2)$$

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$$\begin{aligned} & x_0^{j_1}x_0^{j_2}\ldots x_0^{j_{n-1}}x_0^{j_n} \\ & = (x_0^{j_1}x_0^{j_2}\ldots x_0^{j_{n-1}})x_0^{j_n} \\ & = \{G_{i_1 i_2 \ldots i_{n-1}}^{j_1 j_2 \cdots j_{n-1}}(t)x_t^{i_1}x_t^{i_2}\ldots x_t^{i_{n-1}} \\ & \quad + G_{i_1 i_2 \ldots i_n}^{j_1 j_2 \cdots j_{n-1}}(t)x_t^{i_1}x_t^{i_2}\ldots x_t^{i_n} + \cdots\} \\ & \quad \{S_{i_1}^{j_n}(t)x_t^{i_1} + S_{i_1 i_2}^{j_n}(t)x_t^{i_1}x_t^{i_2} \\ & \quad + S_{i_1 i_2 i_3}^{j_n}(t)x_t^{i_1}x_t^{i_2}x_t^{i_3} + \cdots \\ & \quad + S_{i_1 i_2 \ldots i_n}^{j_n}(t)x_t^{i_1}x_t^{i_2}\ldots x_t^{i_n} + \cdots\} \\ & = G_{i_1 i_2 \ldots i_{n-1}}^{j_1 j_2 \cdots j_{n-1}}(t)S_{i_n}^{j_n}(t)x_t^{i_1}x_t^{i_2}\ldots x_t^{i_n} \\ & \quad + \cdots \\ & = G_{i_1 i_2 \ldots i_n}^{j_1 j_2 \cdots j_n}(t)x_t^{i_1}x_t^{i_2}\ldots x_t^{i_n} \\ & \quad + \cdots \end{aligned} \qquad (7-3)$$

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where, the coefficients $G_{i_1 i_2 \ldots i_n}^{j_1 j_2 \cdots j_m}(t)$ with $m \leq n$ are of the following form:

$$\left\{ \begin{aligned} G_{i_1 i_2 \ldots i_n}^{j_1 j_2}(t) = \; & S_{i_1}^{j_1}(t)S_{i_2 i_3 \ldots i_n}^{j_2}(t) \\ & + S_{i_1 i_2}^{j_1}(t)S_{i_3 i_4 \ldots i_n}^{j_2}(t) + \cdots \\ & + S_{i_1 i_2 \ldots i_{n-2}}^{j_1}(t)S_{i_{n-1} i_n}^{j_2}(t) \\ & + S_{i_1 i_2 \ldots i_{n-1}}^{j_1}(t)S_{i_n}^{j_2}(t) \\ G_{i_1 i_2 \ldots i_n}^{j_1 j_2 j_3}(t) = \; & G_{i_1 i_2 \ldots i_{n-1}}^{j_1 j_2}(t)S_{i_n}^{j_3}(t) \\ & + G_{i_1 i_2 \ldots i_{n-2}}^{j_1 j_2}(t)S_{i_{n-1} i_n}^{j_3}(t) + \cdots \\ & + G_{i_1 i_2}^{j_1 j_2}(t)S_{i_3 i_4 \ldots i_n}^{j_3}(t) \\ \ldots\ldots & \\ G_{i_1 i_2 \ldots i_n}^{j_1 j_2 \cdots j_{n-1}}(t) = \; & G_{i_1 i_2 \ldots i_{n-1}}^{j_1 j_2 \cdots j_{n-2}}(t)S_{i_n}^{j_{n-1}}(t) \\ & + G_{i_1 i_2 \ldots i_{n-2}}^{j_1 j_2 \cdots j_{n-2}}(t)S_{i_{n-1} i_n}^{j_{n-1}}(t) \\ G_{i_1 i_2 \ldots i_n}^{j_1 j_2 \cdots j_n}(t) = \; & G_{i_1 i_2 \ldots i_{n-1}}^{j_1 j_2 \cdots j_{n-1}}(t)S_{i_n}^{j_n}(t) \ . \end{aligned} \right. \qquad (8)$$

Substitution of Eqs. 7-1, Eq. 7-2 and Eq. 7-3 into Eq. 4 gives

$$
\begin{aligned}
x_t^i = \; & \delta^i_{i_1} x_t^{i_1} \\
= \; & T^i_{j_1}(t)\{S^{j_1}_{i_1}(t)x_t^{i_1} + S^{j_1}_{i_1 i_2}(t)x_t^{i_1}x_t^{i_2} \\
& + S^{j_1}_{i_1 i_2 i_3}(t)x_t^{i_1}x_t^{i_2}x_t^{i_3} + \cdots \\
& + S^{j_1}_{i_1 i_2 \ldots i_n}(t)x_t^{i_1}x_t^{i_2}\ldots x_t^{i_n} + \cdots\} \\
& + T^i_{j_1 j_2}(t)\{G^{j_1 j_2}_{i_1 i_2}(t)x_t^{i_1}x_t^{i_2} \\
& + G^{j_1 j_2}_{i_1 i_2 i_3}(t)x_t^{i_1}x_t^{i_2}x_t^{i_3} + \cdots \\
& + G^{j_1 j_2}_{i_1 i_2 \ldots i_n}(t)x_t^{i_1}x_t^{i_2}\ldots x_t^{i_n} + \cdots\} \\
& + \cdots\cdots \\
& + T^i_{j_1 j_2 \ldots j_n}(t)\{G^{j_1 j_2 \ldots j_n}_{i_1 i_2 \ldots i_n}(t)x_t^{i_1}x_t^{i_2}\ldots x_t^{i_n} \\
& + \cdots\} \\
& + \cdots\cdots \quad .
\end{aligned}
\tag{9}
$$

Combination and comparision of the coefficients in Eq.9 yield the following coefficient equations:

$$
\left\{
\begin{aligned}
& T^i_{j_1}(t)S^{j_1}_{i_1}(t) = \delta^i_{i_1} \\
& T^i_{j_1}(t)S^{j_1}_{i_1 i_2}(t) + T^i_{j_1 j_2}(t)G^{j_1 j_2}_{i_1 i_2}(t) = 0 \\
& \cdots\cdots \\
& T^i_{j_1}(t)S^{j_1}_{i_1 i_2 \ldots i_n}(t) + T^i_{j_1 j_2}(t)G^{j_1 j_2}_{i_1 i_2 \ldots i_n}(t) \\
& \quad + T^i_{j_1 j_2 j_3}(t)G^{j_1 j_2 j_3}_{i_1 i_2 \ldots i_n}(t) + \cdots \\
& \quad + T^i_{j_1 j_2 \ldots j_{n-2}}(t)G^{j_1 j_2 \ldots j_{n-2}}_{i_1 i_2 \ldots i_n}(t) \\
& \quad + T^i_{j_1 j_2 \ldots j_{n-1}}(t)G^{j_1 j_2 \ldots j_{n-1}}_{i_1 i_2 \ldots i_n}(t) \\
& \quad + T^i_{j_1 j_2 \ldots j_n}(t)G^{j_1 j_2 \ldots j_n}_{i_1 i_2 \ldots i_n}(t) = 0 \\
& \cdots\cdots \quad .
\end{aligned}
\right.
\tag{10}
$$

which, in turn, yield the coefficients of the inverse transform Eq.5:

$$
\left\{
\begin{aligned}
& S^j_{i_1}(t) = [T^{-1}(t)]^j_i \delta^i_{i_1} \\
& S^j_{i_1 i_2}(t) = -[T^{-1}(t)]^j_i T^i_{j_1 j_2}(t)G^{j_1 j_2}_{i_1 i_2}(t) \\
& \cdots\cdots \\
& S^j_{i_1 i_2 \ldots i_n}(t) = -[T^{-1}(t)]^j_i \{T^i_{j_1 j_2}(t)G^{j_1 j_2}_{i_1 i_2 \ldots i_n}(t) \\
& \qquad + T^i_{j_1 j_2 j_3}(t)G^{j_1 j_2 j_3}_{i_1 i_2 \ldots i_n}(t) + \cdots \\
& \qquad + T^i_{j_1 j_2 \ldots j_{n-2}}(t)G^{j_1 j_2 \ldots j_{n-2}}_{i_1 i_2 \ldots i_n}(t) \\
& \qquad + T^i_{j_1 j_2 \ldots j_{n-1}}(t)G^{j_1 j_2 \ldots j_{n-1}}_{i_1 i_2 \ldots i_n}(t) \\
& \qquad + T^i_{j_1 j_2 \ldots j_n}(t)G^{j_1 j_2 \ldots j_n}_{i_1 i_2 \ldots i_n}(t)\} \\
& \cdots\cdots \quad .
\end{aligned}
\right.
\tag{11}
$$

Arranging above equation with Eq.8, obtain the coefficients of Eq.5 as the following formulae:

$$S^j_{i_1}(t) = [T^{-1}(t)]^j_{i_1} \tag{12 – 1}$$

$$
\begin{aligned}
S^j_{i_1 i_2}(t) = \; & -[T^{-1}(t)]^j_i \\
& T^i_{j_1 j_2}(t)S^{j_1}_{i_1}(t)S^{j_2}_{i_2}(t)
\end{aligned}
\tag{12 – 2}
$$

$$
\begin{aligned}
S^j_{i_1 i_2 i_3}(t) = \; & -[T^{-1}(t)]^j_i \\
& \{2T^i_{j_1 j_2}(t)S^{j_1}_{i_1}(t)S^{j_2}_{i_2 i_3}(t) \\
& + T^i_{j_1 j_2 j_3}(t)S^{j_1}_{i_1}(t)S^{j_2}_{i_2}(t)S^{j_3}_{i_3}(t)\}
\end{aligned}
\tag{12 – 3}
$$

$$
\begin{aligned}
& S^j_{i_1 i_2 i_3 i_4}(t) \\
& \quad = -[T^{-1}(t)]^j_i \{2T^i_{j_1 j_2}(t) \\
& \quad [S^{j_1}_{i_1}(t)S^{j_2}_{i_2 i_3 i_4}(t) + \tfrac{1}{2}S^{j_1}_{i_1 i_2}(t)S^{j_2}_{i_3 i_4}(t)] \\
& \quad + 3T^i_{j_1 j_2 j_3}(t)S^{j_1}_{i_1}(t)S^{j_2}_{i_2}(t)S^{j_3}_{i_3 i_4}(t) \\
& \quad + T^i_{j_1 j_2 j_3 j_4}(t)S^{j_1}_{i_1}(t)S^{j_2}_{i_2}(t)S^{j_3}_{i_3}(t)S^{j_4}_{i_4}(t)\}
\end{aligned}
\tag{12 – 4}
$$

$$
\begin{aligned}
& S^j_{i_1 i_2 i_3 i_4 i_5}(t) = \\
& \quad -[T^{-1}(t)]^j_i \{2T^i_{j_1 j_2}(t) \\
& \quad [S^{j_1}_{i_1}(t)S^{j_2}_{i_2 i_3 i_4 i_5}(t) + S^{j_1}_{i_1 i_2}(t)S^{j_2}_{i_3 i_4 i_5}(t)] \\
& \quad + 3T^i_{j_1 j_2 j_3}(t)[S^{j_1}_{i_1}(t)S^{j_2}_{i_2}(t)S^{j_3}_{i_3 i_4 i_5}(t) \\
& \quad + S^{j_1}_{i_1}(t)S^{j_2}_{i_2 i_3}(t)S^{j_3}_{i_4 i_5}(t)] \\
& \quad + 4T^i_{j_1 j_2 j_3 j_4}(t)S^{j_1}_{i_1}(t)S^{j_2}_{i_2}(t)S^{j_3}_{i_3}(t)S^{j_4}_{i_4 i_5}(t) \\
& \quad + T^i_{j_1 j_2 j_3 j_4 j_5}(t)S^{j_1}_{i_1}(t)S^{j_2}_{i_2}(t)S^{j_3}_{i_3}(t) \\
& \qquad S^{j_4}_{i_4}(t)S^{j_5}_{i_5}(t)\}
\end{aligned}
\tag{12 – 5}
$$

$\cdots\cdots \quad .$

Carrying out the recursive process, the coefficients in the inverse transform Eq.5 are finally expressed in terms of those in the system solution Eq.4:

$$S^j_{i_1}(t) = [T^{-1}(t)]^j_{i_1} \tag{13 – 1}$$

$$
\begin{aligned}
S^j_{i_1 i_2}(t) = \; & -[T^{-1}(t)]^j_i \\
& T^i_{j_1 j_2}(t)[T^{-1}(t)]^{j_1}_{i_1}[T^{-1}(t)]^{j_2}_{i_2}
\end{aligned}
\tag{13 – 2}
$$

$$
\begin{aligned}
S^j_{i_1 i_2 i_3}(t) = \; & -[T^{-1}(t)]^j_i \{-2T^i_{j_1 j_2}(t) \\
& [T^{-1}(t)]^{j_1}_{i_1}[T^{-1}(t)]^{j_2}_{i'_2} \\
& T^{i'_2}_{j'_1 j'_2}[T^{-1}(t)]^{j'_1}_{i_2}[T^{-1}(t)]^{j'_2}_{i_3}] \\
& + T^i_{j_1 j_2 j_3}(t) \\
& [T^{-1}(t)]^{j_1}_{i_1}[T^{-1}(t)]^{j_2}_{i_2}[T^{-1}(t)]^{j_3}_{i_3}\}
\end{aligned}
\tag{13 – 3}
$$

$$
\begin{aligned}
& S^j_{i_1 i_2 i_3 i_4}(t) \\
& \quad = -[T^{-1}(t)]^j_i \{-2T^i_{j_1 j_2}(t) \\
& \quad [[T^{-1}(t)]^{j_1}_{i_1}[T^{-1}(t)]^{j_2}_{i'_2} \\
& \quad (-2T^{i'_2}_{j'_1 j'_2}(t)[T^{j'_1}_{i_2}[T^{-1}(t)]^{j'_2}_{i'_3} \\
& \qquad T^{i'_3}_{k_1 k_2}(t)[T^{-1}(t)]^{k_1}_{i_3}T^{-1}(t)]^{k_2}_{i_4} \\
& \quad + T^{i'_2}_{j'_1 j'_2 j'_3}(t)[T^{-1}(t)]^{j'_1}_{i_2}[T^{-1}(t)]^{j'_2}_{i_3}[T^{-1}(t)]^{j'_3}_{i_4}) \\
& \quad + \tfrac{1}{2}[T^{-1}(t)]^{j_1}_{i'_1}T^{i'_1}_{j'_1 j'_2}(t)[T^{-1}(t)]^{j'_1}_{i_1}[T^{-1}(t)]^{j'_2}_{i_2} \\
& \qquad [T^{-1}(t)]^{j_2}_{i'_3}T^{i'_3}_{j'_3 j'_4}(t)[T^{-1}(t)]^{j'_3}_{i_3}[T^{-1}(t)]^{j'_4}_{i_4}] \\
& \quad - 3T^i_{j_1 j_2 j_3}(t)[T^{-1}(t)]^{j_1}_{i_1}[T^{-1}(t)]^{j_2}_{i_2}[T^{-1}(t)]^{j_3}_{i'_3} \\
& \qquad T^{i'_3}_{j'_1 j'_2}(t)[T^{-1}(t)]^{j'_1}_{i_3}[T^{-1}(t)]^{j'_2}_{i_4} \\
& \quad + T^i_{j_1 j_2 j_3 j_4}(t)[T^{-1}(t)]^{j_1}_{i_1}[T^{-1}(t)]_{i_2}{}^{j_2} \\
& \qquad [T^{-1}(t)]^{j_3}_{i_3}[T^{-1}(t)]^{j_4}_{i_4}\}
\end{aligned}
\tag{13 – 4}
$$

$$
\begin{aligned}
S^{j}_{i_1 i_2 i_3 i_4 i_5}(t) = & -[T^{-1}(t)]^{j}_{i} \\
& \{-2T^{i}_{j_1 j_2}(t)\{[T^{-1}(t)]^{j_1}_{i_1}[T^{-1}(t)]^{j_2}_{i'_2} \\
& \{-2T^{i'_2}_{j'_1 j'_2}(t)[[T^{-1}(t)]^{j'_1}_{i_2}[T^{-1}(t)]^{j'_2}_{i'_3} \\
& \quad (-2T^{i'_3}_{k'_1 k'_2}(t)[T^{-1}(t)]^{k'_1}_{i_3}[T^{-1}(t)]^{k'_2}_{i'_4}, \\
& \quad T^{i'_4}_{k'_1 k'_2}(t)[T^{-1}(t)]^{k'_1}_{i_4}[T^{-1}(t)]^{k'_2}_{i_5} \\
& \quad +T^{i'_3}_{k_1 k_2 k_3}(t)[T^{-1}(t)]^{k_1}_{i_3}[T^{-1}(t)]^{k_2}_{i_4}[T^{-1}(t)]^{k_3}_{i_5}) \\
& \quad +\tfrac{1}{2}[T^{-1}(t)]^{j'_1}_{i'_2}T^{i'_2}_{k'_1 k'_2}(t)[T^{-1}(t)]^{k'_1}_{i_2}[T^{-1}(t)]^{k'_2}_{i_3} \\
& \quad [T^{-1}(t)]^{j'_2}_{i'_4}T^{i'_4}_{k'_1 k'_2}(t)[T^{-1}(t)]^{k'_1}_{i_4}[T^{-1}(t)]^{k'_2}_{i_5}] \\
& \quad -3T^{i'_2}_{j'_1 j'_2 j'_3}(t)[T^{-1}(t)]^{j'_1}_{i_2}[T^{-1}(t)]^{j'_2}_{i_3}[T^{-1}(t)]^{j'_3}_{i'_4} \\
& \quad T^{i'_4}_{k_1 k_2}(t)[T^{-1}(t)]^{k_1}_{i_4}[T^{-1}(t)]^{k_2}_{i_5} \\
& \quad +T^{i'_2}_{j'_1 j'_2 j'_3 j'_4}(t)[T^{-1}(t)]^{j'_1}_{i_2}[T^{-1}(t)]^{j'_2}_{i_3} \\
& \quad [T^{-1}(t)]^{j'_3}_{i_4}[T^{-1}(t)]^{j'_4}_{i_5}\} \\
& +[T^{-1}(t)]^{j_1}_{i'_1}T^{i'_1}_{j'_1 j'_2}(t)[T^{-1}(t)]^{j'_1}_{i_1}[T^{-1}(t)]^{j'_2}_{i_2} \\
& [T^{-1}(t)]^{j_2}_{i'_3}[-2T^{i'_3}_{j'_1 j'_2}(t)[T^{-1}(t)]^{j'_1}_{i_3}[T^{-1}(t)]^{j'_2}_{i'_4} \\
& \quad T^{i'_4}_{k_1 k_2}(t)[T^{-1}(t)]^{k_1}_{i_4}[T^{-1}(t)]^{k_2}_{i_5} \\
& \quad +T^{i'_3}_{j'_1 j'_2 j'_3}(t)[T^{-1}(t)]^{j'_1}_{i_3}[T^{-1}(t)]^{j'_2}_{i_4}[T^{-1}(t)]^{j'_3}_{i_5}]\} \\
& -3T^{i}_{j_1 j_2 j_3}(t)\{[T^{-1}(t)]^{j_1}_{i_1}[T^{-1}(t)]^{j_2}_{i_2}[T^{-1}(t)]^{j_3}_{i'_3} \\
& \quad [-2T^{i'_3}_{j'_1 j'_2}(t)[T^{-1}(t)]^{j'_1}_{i_3}[T^{-1}(t)]^{j'_2}_{i'_4} \\
& \quad T^{i'_4}_{k_1 k_2}(t)[T^{-1}(t)]^{k_1}_{i_4}[T^{-1}(t)]^{k_2}_{i_5} \\
& \quad +T^{i'_3}_{j'_1 j'_2 j'_3}(t)[T^{-1}(t)]^{j'_1}_{i_3}[T^{-1}(t)]^{j'_2}_{i_4}[T^{-1}(t)]^{j'_3}_{i_5}] \\
& \quad +[T^{-1}(t)]^{j_1}_{i_1}[T^{-1}(t)]^{j_2}_{i'_2}T^{i'_2}_{j'_1 j'_2}(t) \\
& \quad [T^{-1}(t)]^{j'_1}_{i_2}[T^{-1}(t)]^{j'_2}_{i_3} \\
& \quad [T^{-1}(t)]^{j_3}_{i'_4}T^{i'_4}_{j'_1 j'_2}(t)[T^{-1}(t)]^{j'_1}_{i_4}[T^{-1}(t)]^{j'_2}_{i_5}\} \\
& -4T^{i}_{j_1 j_2 j_3 j_4}(t)[T^{-1}(t)]^{j_1}_{i_1}[T^{-1}(t)]^{j_2}_{i_2}[T^{-1}(t)]^{j_3}_{i_3} \\
& \quad [T^{-1}(t)]^{j_4}_{i'_4}T^{i'_4}_{j'_1 j'_2}(t)[T^{-1}(t)]^{j'_1}_{i_4}[T^{-1}(t)]^{j'_2}_{i_5} \\
& +T^{i}_{j_1 j_2 j_3 j_4 j_5}(t)[T^{-1}(t)]^{j_1}_{i_1}[T^{-1}(t)]^{j_2}_{i_2}[T^{-1}(t)]^{j_3}_{i_3} \\
& \quad [T^{-1}(t)]^{j_4}_{i_4}[T^{-1}(t)]^{j_5}_{i_5}\}
\end{aligned}
\tag{13-5}
$$

....

3. NONLINEAR TRANSPORT FOR THE PHASE SPACE OF DYNAMIC SYSTEMS

Substitution of the inverse transform Eq.5 into the boundary Eq.1 at the initial state $X_0 = X(t_0)$ to get the boundary Eq.2 at the final state $X_t = X(t)$ completes theoretically the nonlinear transport for the phase space of a general nonlinear dynamic system. As both the system solution Eq.4 $X(t) = X_t(X_0, t)$ and its inverse transform Eq.5 $X(t_0) = X_0(X_t, t)$ are nonlinear, the phase space transport of a general nonlinear dynamic system is a nonlinear process.

As a typical example, in accelerator, the nonlinear beam phase space transport reveals essential differences from the traditional linear beam phase space transport. In Hamiltonian linear transport system, if the initial boundary equation of the accelerator beam phase space is an ellipse[2]:

$$X_0^T \sigma_0^{-1} X_0 = 1 \ . \tag{14}$$

then, it evolves through any time still to an ellipse with its area unchanged:

$$X_t^T \sigma_t^{-1} X_t = 1 \ . \tag{15}$$

where, $|\sigma_t| = |\sigma_0|$.

But in Hamiltonian nonlinear transport system, the substitution of Eq.5 into Eq.14 gives a equation with infinite number of power terms, showing that the beam phase space boundary is sophistically distorted but still with its volume conserved through Hamiltonian nonlinear transport system.

4. DISCUSSION

Based on the above developed theory, an expert system can be implemented on IBM 286 to output automatically the analytical expressions of any order coefficients of the inverse transform Eq.5 for a general dynamic system Eq.(3), as well as the numerical formulation of any order coeffients of the inverse transform Eq.5 for a concrete given nonlinear dynamic system. Finally the expert system demonstrates the evolution process of the phase space boundary from a given initial state to the final state at any time through a given Hamiltonian nonlinear transport system. All these works will be published in other articles.

5. REFERENCES

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