# EMITTANCE GROWTH IN NONLINEAR BEAM GUIDING AND FOCUSING ELEMENTS

W.J.G.M. Kleeven, J.L. Delhez, J.I.M. Botman, G.A. Webers, C.J. Timmermans and H.L. Hagedoorn
Eindhoven Univ. of Technology, Cyclotron Lab., P.O. Box 513, 5600 MB Eindhoven, Netherlands

## ABSTRACT

An analytical calculation is presented of rms emittance growth. The main assumption made is, that the beam divergence is small so that a thin lense approximation can be used when integrating the beam through the optical element. General results are obtained for an optical element with dipole, quadrupole, sextupole and octupole field components. Even higher order multipoles can easily be included. The effect of momentum spread in the beam on emittance is also considered. The results are applicable to all kinds of beam guiding systems, including cyclotron injection lines. As an example, a special azimuthally varying racetrack microtron dipole magnet is considered.

## 1. INTRODUCTION

It is well known that the emittance of a beam, defined as the phase space area divided by $\pi$, is conserved. However, when the phase space deviates considerably from an elliptical shape, then the phase space area is not such a useful quantity to express the quality of the beam. A more practical quantity is the so-called root-mean-square (rms) emittance. The rms emittance is not always conserved. We will consider two sources of rms emittance growth namely nonlinearities in the beam guiding system and momentum spread in the beam. In the first case the emittance growth is caused by a nonlinear deformation of the phase space due to the fact that the transverse kick experienced by a particle is not proportional to its transverse position. In the second case the emittance growth is due to the fact that an off-momentum particle experiences a deviating bending angle and/or focusing force as compared to the reference momentum. The different sources of emittance increase are illustrated in Fig. 1.

## 2. ROOT MEAN SQUARE PROPERTIES OF THE BEAM

We consider the transverse phase space of the beam with the variables $\boldsymbol{x} = (x, p_x, y, p_y)$, where $x$ and $y$ are the spatial coordinates and $p_x$ and $p_y$ the divergences, i.e. the transverse kinetic momenta, scaled with respect to the total kinetic momentum of the particle. The rms sizes $\widetilde{x}$ and $\widetilde{y}$ of the beam are defined by $\widetilde{x} = \langle x^2 \rangle^{1/2}$ and $\widetilde{y} = \langle y^2 \rangle^{1/2}$ where the brackets denote averaging over the phase space. Note that for an unbunched beam with uniform elliptical cross-section the actual beam size is twice the rms value. The rms emittance in the $x$-$p_x$ phase plane is defined as

$$\varepsilon_x = \left(\langle x^2 \rangle \langle p_x^2 \rangle - \langle x p_x \rangle^2\right)^{1/2}. \qquad (1)$$

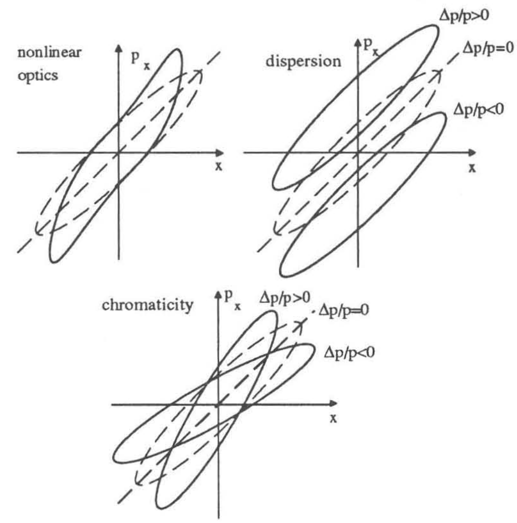


Figure 1. *A graphical illustration of three causes for emittance growth*

A similar definition holds for the $y$-$p_y$ phase plane. To relate this definition with the phase space area occupied by the beam, we assume that the projected distributions have elliptical symmetry. For example in the $x$-$p_x$ phase plane

$$f(x, p_x) = f(U) \quad \text{where} \quad U = cx^2 + 2axp_x + bp_x^2, \qquad (2)$$

with $a$, $b$ and $c$ arbitrary constants. For such an elliptical distribution, the number of particles $\overline{N}$ that is enclosed by an ellips with area $A = 4\pi\varepsilon_x$, is given by

$$\frac{\overline{N}}{N} = \int_0^{\lambda_{max}} f(\lambda)d\lambda \quad \text{with} \quad \lambda_{max} = 2\int_0^{\infty} \lambda f(\lambda)d\lambda, \qquad (3)$$

where $N$ is the total number of particles and where the normalization $\int_0^{\infty} f(\lambda)d\lambda = 1$ is assumed. In Table 1 the values of $\overline{N}/N$ are given for five distributions. A definition of these distributions has been given by Sacherer.[1] For a uniform elliptical phase space distribution the actual area is $4\pi$ times the rms emittance. Therefore, the 'ordinary' emittance is approximately four times the rms emittance.

The beam envelopes $\widetilde{x}$ and $\widetilde{y}$ involve second order spatial moments of the beam. One can also define higher order spatial moments like $\langle x^4 \rangle$, $\langle x^2 y^2 \rangle$, $\langle x^6 \rangle$ etc. If the beam density profile is symmetric with respect to $x$ and $y$, then the spatial moments odd in $x$ or $y$ are zero. The even spatial moments can be expressed in terms of the

Table 1. parameters $\overline{N}/N$, $\xi_1$ and $\xi_2$ as defined in Eqs. (3), and (6), respectively, for five distributions.

| distribution | $\overline{N}/N$ | $\xi_1$ | $\xi_2$ |
|---|---|---|---|
| uniform | 1 | 1 | 1 |
| linear | 0.889 | 1.13 | 1.35 |
| parabolic | 0.914 | 1.07 | 1.19 |
| gaussian | 0.889 | 1.18 | 1.57 |
| hollow | 0.983 | 0.884 | 0.785 |

beam envelopes $\tilde{x}$ and $\tilde{y}$. For later use we calculate the spatial moments up to sixth order. For a beam profile with elliptical symmetry these are

$$\langle x^4\rangle = 2\xi_1\tilde{x}^4, \quad \langle x^2y^2\rangle = \tfrac{2}{3}\xi_1\tilde{x}^2\tilde{y}^2, \tag{4}$$

$$\langle x^6\rangle = 5\xi_2\tilde{x}^6, \quad \langle x^4y^2\rangle = \xi_2\tilde{x}^4\tilde{y}^2, \tag{5}$$

with $\xi_1$ and $\xi_2$ given by

$$\xi_1 = \frac{3}{4}\frac{\int_0^\infty \lambda^2 n(\lambda)d\lambda}{\left(\int_0^\infty \lambda n(\lambda)d\lambda\right)^2}, \quad \xi_2 = \frac{1}{2}\frac{\int_0^\infty \lambda^3 n(\lambda)d\lambda}{\left(\int_0^\infty \lambda n(\lambda)d\lambda\right)^3}, \tag{6}$$

and where $n$ represents the beam density profile with $\int_0^\infty n(\lambda)d\lambda = 1$. The values of $\xi_1$ and $\xi_2$ for five profiles are given in Table 1. We note that the moments $\langle y^4\rangle$, $\langle x^2y^4\rangle$ and $\langle y^6\rangle$ are easily obtained from Eqs. (4) and (5) using symmetry considerations.

## 3. EMITTANCE GROWTH DUE TO NONLINEAR OPTICS

To calculate the emittance growth of the beam we treat the optical element as a nonlinear thin lense. In this assumption the transverse coordinates stay constant when the beam passes through the element and the divergences change by an amount that only depends on the initial transverse displacements. In most cases this approximation is accurate if $L\varepsilon_x/\tilde{x} \ll \tilde{x}$, where $L$ is the length of the optical element. The left-hand-side of this unequality represents the transverse displacement of a particle with divergence $\varepsilon_x/\tilde{x}$, after passing a drift of length $L$. This displacement must be small compared to the beam size.

Within the thin lense approximation the transformation of the phase space through the optical element can be written as

$$\overline{x} = x, \quad \overline{p}_x = p_x + g(x,y), \tag{7}$$

$$\overline{y} = y, \quad \overline{p}_y = p_y + h(x,y), \tag{8}$$

where $x, p_x, y, p_y$ are the initial variables and $\overline{x}, \overline{p}_x, \overline{y}, \overline{p}_y$ the final variables. If the initial and final variables are canonical variables (which is the case for example if the entrance and exit are in a field-free region) then the Jacobian of the transformation must be a symplectic matrix. This then puts the condition $\partial g/\partial y = \partial h/\partial x$ on the functions $g$ and $h$.

We can calculate the new emittance $\overline{\varepsilon}_x$ in the $x$-$p_x$ plane by substitutings Eqs. (7) into Eq. (1). This gives

$$\overline{\varepsilon}_x^2 - \varepsilon_x^2 = \langle x^2\rangle\langle g^2\rangle - \langle xg\rangle^2 + 2\langle x^2\rangle\langle p_x g\rangle - 2\langle xp_x\rangle\langle xg\rangle. \tag{9}$$

This expression simplifies if the initial phase space distribution function (written in terms of the variables $x$ and $p_x - \eta x$) possesses the following symmetry property

$$f(x, p_x - \eta x, y, p_y) = f(x, -p_x + \eta x, y, p_y), \tag{10}$$

where $\eta$ is an arbitrary parameter. This symmetry property is illustrated graphically in Fig. 2. Note that the elliptically symmetric distribution given in Eq. (2) obeys Eq. (10) by choosing $\eta = -a/b$. For distributions obeying Eq. (10), the terms involving $p_x$ in Eq. (9) cancel and the expression for emittance growth becomes

$$\overline{\varepsilon}_x^2 - \varepsilon_x^2 = \langle x^2\rangle\langle g^2\rangle - \langle gx\rangle^2. \tag{11}$$

A similar expression holds for the $y$-$p_y$ phase plane. Since the function $g$ depends on spatial coordinates, the right-hand-side of Eq. (11) only involves spatial moments. These are determined by the density profile of the beam.

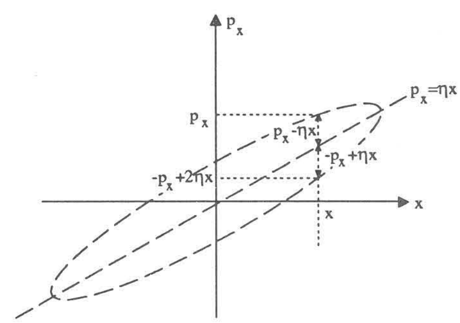


Figure 2. *Graphical illustration of the symmetry property of the phase space distribution function as assumed in the calculation of emittance growth.*

As an example, we calculate the emittance growth for an optical element that has the $y = 0$ plane as a symmetry plane. Then the function $g$ must be even in $y$ and the function $h$ must be odd in $y$. These functions are expanded into a power series with respect to $x$ and $y$. The most general expansion upto third degree is

$$g(x,y) = Ax + Bx^2 + Cy^2 + Dx^3 + Exy^2, \tag{12}$$

$$h(x,y) = Fy + Gxy + Hy^3 + Ix^2y, \tag{13}$$

where the quantities $A, B, \ldots, I$ are arbitrary constants. For a symplectic map they must obey the conditions $G = 2C$ and $I = E$. We insert Eq. (12) into Eq. (11)

and assume that the beam density profile has elliptical symmetry so that Eqs. (4) and (5) apply. The emittance growth formula then becomes

$$\begin{aligned}\bar{\varepsilon}_x^2 - \varepsilon_x^2 &= 2\xi_1[B^2\tilde{x}^6 + \tfrac{2}{3}BC\tilde{x}^4\tilde{y}^2 + C^2\tilde{x}^2\tilde{y}^4] + \\ &+ (5\xi_2 - 4\xi_1^2)D^2\tilde{x}^8 + 2(\xi_2 - \tfrac{4}{3}\xi_1^2)DE\tilde{x}^6\tilde{y}^2 + \\ &+ (\xi_2 - \tfrac{4}{9}\xi_1^2)E^2\tilde{x}^4\tilde{y}^4, \qquad (14)\end{aligned}$$

with the parameters $\xi_1$ and $\xi_2$ defined in Eqs. (6). As can be seen from this expression the emittance growth only depends on the nonlinear parts in the function $g$. This is true in general as can easily be shown from Eq. (11). Further note that there are no cross-correlations between sextupoles and octupoles. It can be shown in general that for an expansion upto arbitrary degree, there are no correlations between odd and even multipoles. The emittance growth in the $y$-direction is calculated in the same way as in the $x$-direction. For this we find

$$\begin{aligned}\bar{\varepsilon}_y^2 - \varepsilon_y^2 &= \tfrac{2}{3}\xi_1 G^2\tilde{x}^2\tilde{y}^4 + (\xi_2 - \tfrac{4}{9}\xi_1^2)I^2\tilde{x}^4\tilde{y}^4 \qquad (15)\\ &+ 2(\xi_2 - \tfrac{4}{3}\xi_1^2)HI\tilde{x}^2\tilde{y}^6 + (5\xi_2 - 4\xi_1^2)H^2\tilde{y}^8.\end{aligned}$$

The results obtained so far will now be further specified for a nonlinear bending magnet.

## 4. EMITTANCE GROWTH IN A BENDING MAGNET

We use the Hamilton formalism to solve the equations of motion for a non-linear bending magnet within the thin-lense approximation. The transverse motion of a particle with respect to reference trajectory lying in the symmetry plane $y = 0$ is described by the Hamiltonian

$$\begin{aligned}H &= -(1+\frac{x}{\rho})\sqrt{P^2 - (P_x - qA_x)^2 - (P_y - qA_y)^2} \\ &\quad -q(1+\frac{x}{\rho})A_s, \qquad (16)\end{aligned}$$

where $\rho$ is the radius of curvature of the reference trajectory, $P$ is the kinetic momentum of the particle, $P_x$ and $P_y$ are the transverse canonical momenta and $A_x, A_y, A_s$ are the components of the magnetic vector potential. The independent variable $s$ is the distance along the reference trajectory. The scaled kinetic momenta $p_x, p_y$ as introduced in section 2 are related to the canonical momenta $P_x, P_y$ by the equations

$$p_x = \frac{P_x - qA_x}{P}, \qquad p_y = \frac{P_y - qA_y}{P}. \qquad (17)$$

In order to obtain equations of motion for the divergences $p_x$ and $p_y$ we first derive the canonical equations of motion from the Hamiltonian given in Eq. (16). The right-hand-sides of the canonical equations for $P_x$ and $P_y$ still contain the canonical momenta. These are eliminated in favour of $dx/ds$ and $dy/ds$ by using the canonical equations for $x$ and $y$. Then the equations for $dp_x/ds$ and $dp_y/ds$ are obtained by differentiating Eqs. (17) with respect to $s$. The resulting equations express $dp_x/ds$ and $dp_y/ds$ in terms of $x$, $y$, $dx/ds$ and $dy/ds$. Since in the thin-lense approximation the transverse coordinates stay constant, we can insert into these equations $dx/ds = 0$ and $dy/ds = 0$. The final result is then

$$\begin{aligned}\frac{dp_x}{ds} &= \frac{1}{\rho} + \frac{q}{P}\left[\frac{A_s}{\rho} - \frac{\partial A_x}{\partial s} + (1+\frac{x}{\rho})\frac{\partial A_s}{\partial x}\right], \qquad (18)\\ \frac{dp_y}{ds} &= \frac{q}{P}(1+\frac{x}{\rho})\frac{\partial A_s}{\partial y}. \qquad (19)\end{aligned}$$

Expansions of the magnetic vector potential upto fourth degree have been given by Corsten.[2)] The phase space map is obtained by inserting these expansions into Eqs. (18) and (19) and then integrating the particle through the magnet. Assuming that the entrance and exit are in a field-free region we get *

$$\begin{aligned}\bar{p}_x &= p_x - (\alpha + \bar{\alpha})x - \tfrac{1}{2}(\beta + 2\bar{\beta})x^2 + \tfrac{1}{2}(\beta + \bar{\beta})y^2 \\ &- \tfrac{1}{6}(\gamma + \tfrac{5}{2}\bar{\gamma} + \tfrac{1}{2}\bar{\delta})x^3 + \tfrac{1}{2}(\gamma + \tfrac{3}{2}\bar{\gamma} + \tfrac{1}{2}\bar{\delta})xy^2, (20)\\ \bar{p}_y &= p_y + \alpha y + (\beta + \bar{\beta})xy - \tfrac{1}{6}(\gamma + \tfrac{1}{2}\bar{\gamma} - \tfrac{1}{2}\bar{\delta})y^3 \\ &+ \tfrac{1}{2}(\gamma + \tfrac{3}{2}\bar{\gamma} + \tfrac{1}{2}\bar{\delta})x^2 y, \qquad (21)\end{aligned}$$

with the definitions

$$\begin{aligned}\alpha &= \frac{q}{P}\int\left(\frac{\partial B_y}{\partial x}\right)_0 ds, \quad \bar{\alpha} = \frac{q}{P}\int\frac{B_0}{\rho}ds,\\ \beta &= \frac{q}{P}\int\left(\frac{\partial^2 B_y}{\partial x^2}\right)_0 ds, \quad \bar{\beta} = \frac{q}{P}\int\frac{1}{\rho}\left(\frac{\partial B_y}{\partial x}\right)_0 ds,\\ \gamma &= \frac{q}{P}\int\left(\frac{\partial^3 B_y}{\partial x^3}\right)_0 ds, \quad \bar{\gamma} = \frac{q}{P}\int\frac{1}{\rho}\left(\frac{\partial^2 B_y}{\partial x^2}\right)_0 ds,\\ \bar{\delta} &= \frac{q}{P}\int\left(\frac{B_0''}{\rho} + \frac{1}{\rho^2}\left(\frac{\partial B_y}{\partial x}\right)_0\right) ds, \qquad (22)\end{aligned}$$

where the integration runs from the entrance to the exit of the magnet and where the prime denotes differentiation with respect to $s$. The quantity $B_0$ is defined as $B_0 = B_y(x = y = 0)$. The map through the magnet is symplectic. A comparison of Eqs. (20) and (21) with Eqs. (12) and (13) gives the relations between the parameters $A, B, \ldots, I$ as introduced in section 3 and the field quantities $\alpha, \beta, \ldots, \bar{\delta}$.

For a pure sextupole (only $\beta \neq 0$) and a round input beam ($\tilde{x} = \tilde{y}$) the emittance growth is given by

$$\bar{\varepsilon}_x^2 - \varepsilon_x^2 = \bar{\varepsilon}_y^2 - \varepsilon_y^2 = \tfrac{2}{3}\xi_1\beta^2\tilde{x}^6. \qquad (23)$$

So, in order that the effects are small we must have $\beta \ll \varepsilon/\tilde{x}^3$. If the input emittances are upright then this can be written as $\beta \ll \tilde{p}_x/\tilde{x}^2$ where $\tilde{p}_x = \langle p_x^2\rangle^{1/2}$ is the rms divergence of the beam. For a pure octupole (only $\gamma \neq 0$) and a round input beam the emittance growth is given by

$$\bar{\varepsilon}_x^2 - \varepsilon_x^2 = \bar{\varepsilon}_y^2 - \varepsilon_y^2 = \tfrac{2}{9}\xi_2\gamma^2\tilde{x}^8. \qquad (24)$$

*A sufficient assumption would be that at the entrance and exit first order derivatives of the field quantities with respect to $s$ are zero, as is the case for example in a symmetry point of the magnet.

## 5. THE INFLUENCE OF MOMENTUM SPREAD

So far, a mono-energetic input beam was assumed. Momentum spread gives additional emittance growth, even if the optics in the bending magnet is linear. If the deviating momentum is $\delta = \Delta P/P$, then the functions $g$ and $h$ defined by Eqs. (7), (8), (20) and (21) should be replaced by functions $\overline{g}$ and $\overline{h}$ as follows:

$$\overline{g} = \frac{\delta\Delta\phi + g}{1+\delta}, \quad \overline{h} = \frac{h}{1+\delta}, \tag{25}$$

where $\Delta\phi$ is the total bending angle of the reference particle. To calculate the additional emittance growth it is convenient to assume that the distribution functions for the momentum spread and the transverse phase space are uncorrelated. Furthermore, since $\delta$ is small, only the most significant terms in $\delta$ and only the linear parts of the functions $g$ and $h$ are taken into account. Then the additional emittance growth due to momentum spread is

$$\overline{\varepsilon}_x^2 - \varepsilon_x^2 = \widetilde{\delta}^2\widetilde{x}^2[\Delta\phi^2 + (\alpha+\overline{\alpha})^2\widetilde{x}^2], \quad \overline{\varepsilon}_y^2 - \varepsilon_y^2 = \alpha^2\widetilde{\delta}^2\widetilde{y}^4, \tag{26}$$

where $\widetilde{\delta} = \langle\delta^2\rangle^{1/2}$ is the rms momentum spread and with $\alpha$ and $\overline{\alpha}$ defined in Eqs. (22).

## 6. AN EXAMPLE

As an application we consider the case of a racetrack microtron dipole magnet with an azimuthally varying median plane field. A microtron with such magnets is currently under construction in our laboratory.[3] In general, we may write the field map of such a magnet in polar coordinates $(r, \theta)$ as follows:

$$B_y(r,\theta) = \widehat{B}_y[1 + f(\theta)], \quad 0 \le \theta \le \tfrac{1}{2}\pi, \tag{27}$$

with $\widehat{B}_y$ the nominal field strength and $f(\theta)$ defining the (small) perturbing field profile. The orbits through this magnet start at $r = \theta = 0$ and it is assumed that the deviating field still provides for a 180° bend.

The Hamilton formalism has been applied to obtain an expression for the reference orbit through the above field map. From this we can derive expressions for the field gradients as a function of orbit length $s$. Since $f(\theta)$ is assumed to be small, all calculations are done first order in $f$. The coefficients $\alpha, \beta, \ldots, \overline{\delta}$ are calculated using Eq. (22). From these, the parameters $A, B, \ldots, I$ can be found and hence the emittance growth as described by the Eqs. (14) and (15). For a round beam we may write

$$\overline{\varepsilon}_x^2 - \varepsilon_x^2 = S_x\widetilde{x}^6 + O_x\widetilde{x}^8, \quad \overline{\varepsilon}_y^2 - \varepsilon_y^2 = S_y\widetilde{x}^6 + O_y\widetilde{x}^8, \tag{28}$$

where $S_x, O_x, S_y, O_y$ define the sextupole and octupole contributions to the emittance growth in the $x$ and $y$ directions. As an example, we take the field profile

$$f(\theta) = \widehat{f}\sin^{10}(2\theta), \tag{29}$$

describing a smooth hill or valley, centered around $\theta = \frac{1}{4}\pi$ with a height or depth determined by $\widehat{f}$. Assuming a uniform distribution ($\xi_1 = \xi_2 = 1$), we get

$$S_x \approx 9\widehat{f}^2/R^4, \quad O_x \approx 900\widehat{f}^2/R^6, \tag{30}$$

where $R = P/(q\widehat{B}_y)$. The values for the $y$ direction are of the same order of magnitude, so we will only consider the $x$ direction. First of all, we note that the octupole component becomes significant when $\widetilde{x} \approx \sqrt{S_x/O_x} \approx 0.1R$. From this it may be concluded that in practical cases the octupole term can be neglected. Assuming that the emittance growth is small with respect to the initial emittance, we may rewrite Eq. (28) as follows

$$\frac{\Delta\varepsilon_x}{\varepsilon_x} \approx \frac{S_x\widetilde{x}^6}{2\varepsilon_x^2} = \frac{9\widehat{f}^2R^2}{2\varepsilon_x^2}\left(\frac{\widetilde{x}}{R}\right)^6, \tag{31}$$

expressing the relative change of the emittance as caused by the sextupole component only. For a numerical example, we choose values appropriate for our own microtron at injection energy, viz. $\varepsilon_x = 6$ mmmrad, $R = 0.10$ m, and $\widehat{f} = 0.5$. The result is plotted in Fig. 3. As can be seen, the change of emittance is small for $\widetilde{x}/R < 0.02$, or a beam diameter of less than 8 mm. For completeness, the octupole term has been drawn as well, which has an effect that is approximately an order of magnitude smaller.

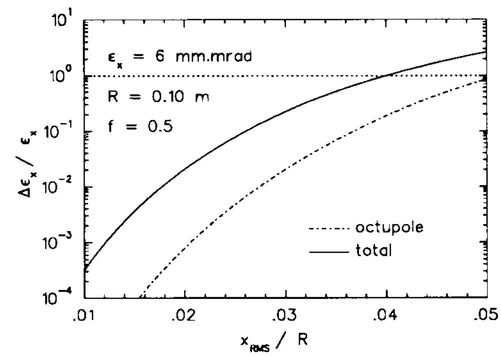


Figure 3. *Relative change of emittance in the x-direction as a function of scaled rms beam radius for a smooth hill or valley field profile.*

## 7. REFERENCES


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