ON THE THEORY OF THE COUPLING BETWEEN RADIAL OSCILLATIONS AND H.F. PHASE AND ENERGY OF PARTICLES IN CYCLOTRONS

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Abstract

The motion of accelerated particles is described by a Hamilton function in which the acceleration is taken into account by a timedependent potential function. By canonical transformations four variables are defined of which one set describes energy and H.F.phase and the other set the radial oscillations of the particles. In the new Hamilton function the coupling between these two sets of variables is clearly demonstrated.

Particles in a cyclotron with a homogeneous field and an one-Dee system and particles in a cyclotron with a four-fold symmetric magnetic field and a two-Dee system are concidered.

1. Introduction

The motion of non-accelerated particles in cyclotrons has been described by several authors [1,2,3]. In this paper an onset of a general analytical orbit theory, which also takes into account the acceleration, will be presented. The purpose of this theory is to give a description of the influence of the acceleration on the beamproperties by relatively simple equations. With these equations a simple computerprogram can be made by which a good understanding of several effects can be obtained. Aside of this analytical approach in the near future a numerical integration procedure will be used to check the results and to acquire final data. The general orbit theory together with the numerical programs will be applied to the four-fold symmetric separated sector cyclotron of the Hahn-Meitner-Institut to find the influence of radial oscillations on the H.F.phase and energy of the particles [4].

The motion of the particles in the median plane is described by a Hamilton function in which cartesian coordinates are used. After several canonical transformations fastly moving parts are separated from slowly moving parts, which represent the wellknown motion of the orbitcentre. In the fastly moving parts energy and H.F. phase are described. Coupling terms between both types of motion have been found.

A treatise of the complete motion of accelerated particles in polar coordinates leads to serious analytical problems.

In section 2 we will discuss the theory of non-accelerated particles and in section 3 the theory of accelerated particles.

2. Non-accelerated particles

2.1 Homogeneous field

The Hamilton function for particles in the median plane, under influence of a magnetic and an electric field is given by:

$$H = \frac{1}{2m}(P_x - eA_x)^2 + \frac{1}{2m}(P_y - eA_y)^2 + eV(x,y,t) \qquad (1)$$

in which P_x and P_y are the canonical impulses in the x- and y-direction; A is the vector potential of the magnetic field B and V is the scalar potential of the electric field; m and e are mass and charge of the particles; t is time.

In this section we only consider the case with V=0. The vector potential for a homogeneous field with magnetic fieldstrength B_0 in the z-direction is represented by:

$$A_x = -B_0 y \ , \ A_y = A_z = 0 \qquad (2)$$

so:

$$H = \frac{1}{2m}(P_x + eB_0 y)^2 + \frac{1}{2m}P_y^2 \qquad (3)$$

For convenience we introduce new variables:

$$\bar{p}_x = \frac{p_x}{m} \ , \ \bar{p}_y = \frac{p_y}{m} \ , \ E = \omega_0 t$$
$$\bar{y} = \omega_0 y \ , \ \bar{x} = \omega_0 x \ , \ \omega_0 = \frac{eB}{m} \qquad (4)$$

This yields:

$$H = \tfrac{1}{2}(\bar{p}_x + \bar{y})^2 + \tfrac{1}{2}\bar{p}_y^2 \qquad (5)$$

The motion is split in a fastly and a slowly oscillating part by the canonical transformation:

$$\bar{x} = \bar{\bar{x}} + \bar{\bar{p}}_y \qquad p_x = -\bar{\bar{y}}$$
$$\bar{y} = \bar{\bar{y}} + \bar{\bar{p}}_x \qquad p_y = -\bar{\bar{x}} \qquad (6)$$

The new Hamiltonfunction then becomes:

$$\bar{\bar{H}} = \tfrac{1}{2}\bar{\bar{p}}_x^2 + \tfrac{1}{2}\bar{\bar{x}}^2 \qquad (7)$$

The $\bar{\bar{p}}_y, \bar{\bar{y}}$ coordinates have disappeared from the Hamilton function. They are therefore constants e.g. C_1 resp. C_2. This corresponds to the position of the centre of the circle, described by the particles. The solution for the $\bar{\bar{x}}, \bar{\bar{p}}_x$-coordinates is the circle motion, which has the frequency 1 (fastly oscillating). In the old coordinates the solution is:

$$x = R_0 \cos \omega_0 (t - t_0) + C_1 \qquad (8)$$
$$y = -R_0 \sin \omega_0 (t - t_0) + C_2$$

R_0, t_0, C_1, C_2 are constants determined by the initial conditions.

For a cylindrically symmetrical field the solution shows clearly again the fast part as given in equation (8). However the slow part now shows an oscillation with frequency $\nu_r - 1$, where ν_r is the radial oscillation frequency and therefore describes again the motion of the orbit centre.

2.2 Azimuthally varying field

The Hamiltonfunction for a particle in an azimuthally varying field $B_z = B_0(1 + A_n \cos n\theta)$ can be represented by:

$$H = \frac{1}{2m}\left(P_x + yB_0 - \frac{B_0 x}{n^2} f_n'\right)^2 + \frac{1}{2m}\left(P_y - \frac{B_0 y}{n^2} f_n'\right)^2$$
$$\text{with } f_n = A_n \cos n\theta \ , \ f_n' = \frac{\delta f_n}{\delta \theta} \qquad (9)$$

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f_n is a function of the azimuth θ, which on its turn is a function of the cartesian coordinates x and y. After application of the transformations (4) and (6) the new Hamiltonfunction becomes:

$$H = \tfrac{1}{2}\left(\bar{\bar{p}}_x - \frac{\bar{\bar{x}}+\bar{\bar{p}}_y}{n^2} f_n'\right)^2 + \tfrac{1}{2}\left(\bar{\bar{x}} + \frac{\bar{\bar{y}}+\bar{\bar{p}}_x}{n^2} f_n'\right)^2, \quad (10)$$

Only the coefficients which are linear in the amplitude A_n are kept. At first sight this means that our treatment will only be valid for small values of A_n. However the results are consistent with those of the general orbit theory for non-accelerated particles in ref.1. As these latter results may be regarded to be valid for $A_n/n \ll 1$ we assume that the treatment, presented here, has the same approximation.[5]

Expansion of the Hamilton function in y and p_y up to second degree (leaving the bars) yields:

$$H = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}x^2 - \frac{1}{n^2}(p_x p_y - xy) f_n' + \frac{f_n''}{n^2}\frac{x^2}{R^2}\tfrac{1}{2}y^2 + \frac{f_n''}{n^2}\frac{p_x^2}{R^2}\tfrac{1}{2}p_y^2 - \frac{f_n''}{n^2}\frac{p_x x}{R^2} p_y y \quad (11)$$

We now want to apply a transformation which yields a Hamilton function in which no couplingterms, which are linear in y, p_y, are present. Then we can find a solution for which $y=p_y=0$. This solution represents the central orbit for one energy. The motion of the orbit centre is found by solving the equation of motion for y, p_y and substituting the central orbit solution.

For n>1 this transformation is given by the generating function $G(x,\bar{p}_x,y,\bar{p}_y)$:

$$G = \bar{p}_x x + \bar{p}_y y + \frac{1}{n^2-1}(xy - \bar{p}_x\bar{p}_y)\bar{f} \quad \frac{1}{n^2(n^2-1)}(x\bar{p}_y + y\bar{p}_x)\bar{f}' \quad (12)$$

where $\bar{f}=f(x,\bar{p}_x)$

Then the real orbit coordinates (x_{geom}, y_{geom}) are given by:

$$y_{geom} = \bar{y} + \frac{1}{n^2-1}\bar{p}_x f_n + \frac{1}{n^2(n^2-1)}\bar{x} f_n' + \bar{p}_x$$

$$x_{geom} = \bar{p}_y + \frac{1}{n^2-1}\bar{x} f_n - \frac{1}{n^2(n^2-1)}\bar{p}_x f_n' + \bar{x} \quad (13)$$

The second term at the right hand side of eq(13) determines the wellknown oscillation of the equilibrium orbit around a circle. The third term at the right hand side ensures that the velocity of the particles is constant along its path. The motion of the orbit centre is represented by $(\bar{y},\bar{p}_y)$ in the new Hamilton function.

The coefficients of $\bar{y}^2, \bar{p}_y^2$ and $\bar{p}_y\,\bar{y}$ are first order oscillating functions for n>2 and can be transformed to higher order [1]. Therefore the oscillation frequency ν_r-1 of $\bar{y}$ is of second order.

The influence of first and second harmonic field pertubations is described by remaining first and second degree terms, which can not be transformed away. This describtion is in accordance with that in ref 6. Further a derivative in the mean magnetic field gives rise to constant coefficients (see section 2.1)

3 Accelerated particles

3.1 Homogeneous field and one-dee system

As an example we consider here a particle in a homogeneous field, accelerated by a one-dee system.

The electrical potential for a one-dee system can be represented by $eV_0 He(y)\cos\omega_{H.F.}t$. V_0 is the amplitude of the acceleration voltage; for simplicity we take He(y) equal to a stepfunction in the y-direction (this does not limit the generality of our treatment); $\omega_{H.F.}$ is the frequency of the acceleration voltage.

After the transformations (4) and (6) the time-dependent Hamilton function becomes:

$$H = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}x^2 + e\bar{V}He(y+p_x)\cos\omega t$$
$$\text{with: } \omega = \omega_{H.F.}/\omega_{particles} \qquad \bar{V} = V/m \quad (14)$$

We define new variables π_x, ξ by $p_x=\pi_x+p_0$ and $x=\xi+x_0$. x_0 and p_0 are the solution for particles which have zero phase and are well-centered. For this solution we require that the first degree terms in the Hamilton function vanish. Then π_x and ξ determine H.F.phase and energy of the particles. The meaning of x_0, p_0, ξ and π_x in the geometric coordinates is illustrated in fig.1a and 1b, in which a particle with a negative phase resp. an energy deviation, is given ($y=p_y=0$). The position of the particle is found by vectorially adding the vectors (x_0,p_0) and (ξ,π_x).

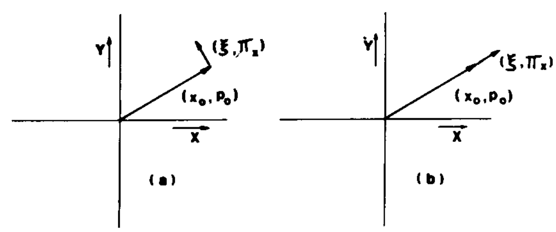

fig.1: *Illustration of the meaning of the variables*

The Hamilton function in the new variables is:

$$H = \tfrac{1}{2}\pi_x^2 + \tfrac{1}{2}\xi^2 + \pi_x\left(p_0 - \frac{x_0}{t}\right) + \xi\left(x_0 + \frac{p_0}{t}\right) + e\bar{V}He(y+\pi_x+p_0)\cos\omega t \quad (15)$$

Again for simplicity we choose $\omega=1$. By making ω time dependent one arrives at the equations which describe particles in a synchro-cyclotron.

A new function F is introduced by $F(y+\pi_x,t) = He(\pi_x+y+p_0) - He(p_0)$. Because $He(p_0)$ is a pure function of the time the stepfunction in (15) can be replaced by F. In fig.2 the function $F(\pi_x+y,t)$ is shown for a constant and positive value of the argument (π_x+y).

The time difference between the time of gap-crossing of a particle on a centered orbit with zero H.F.phase and a particle with a non centered orbit and a non zero phase, is called Δt (see fig.3): Δt is a function of $y+\pi_x$ and t.

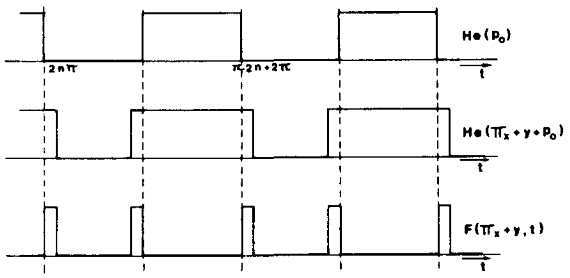

fig.2: *The function $He(p_o)$, $He(\pi_x+y+p_o)$ and $F(\pi_x+y,t)$ for constant π_x+y; n is the number of revolutions from the start at zero velocity.*

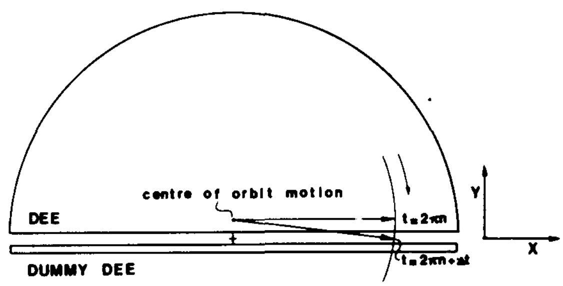

fig.3: *An one-dee system. The timedifference (Δt) of gapcrossing of a well-centered and an off-centered motion is shown.*

We expand F in a fourierseries with coefficients that are functions of t.

$$F = \sum_{n=0}^{\infty} a_{2n}(\Delta t)\cos 2nt + \sum_{n=0}^{\infty} b_{2n+1}(\Delta t)\sin(2n+1)t \qquad (16)$$

$$\text{with: } \Delta t = \frac{\pi_x+y}{\rho t^{\frac{1}{2}}} + \frac{1}{6}\,\frac{(\pi_x+y)^3}{\rho^3 t^{3/2}} + \ldots$$

$\rho t^{\frac{1}{2}}$ corresponds with the radius of a central accelerated orbit, $\rho^2 = 2eV/\pi m$

π_x and y are small quantities. Therefore F can be expanded in a powerseries in π_x and y. The Hamilton function in eq.14 now becomes:

$$\begin{aligned} H = {} & \tfrac{1}{2}\pi_x^2 + \tfrac{1}{2}\xi^2 + \pi_x\left(p_o - \frac{dx_o}{dt}\right) + \xi\left(x_o + \frac{dp_o}{dt}\right) \qquad (17)\\ & + (\pi_x+y)\,\frac{2eV}{\pi\rho t^{\frac{1}{2}}}(\cos t + \cos 3t + ..)\\ & + (\pi_x+y)^2\left(\frac{1}{t}\sin 2t + \frac{2}{t}\sin 4t + ..\right)\\ & + (\pi_x+y)^3\left(-\frac{1}{6\rho t^{3/2}}\cos t - \frac{1}{6\rho t^{3/2}}\cos 3t - ..\right) \end{aligned}$$

As mentioned above x_o and p_o must be choosen so, that the first degree part in π_x and ξ disappeares. We then find:

$$\begin{aligned} p_o &= -\rho t^{\frac{1}{2}}\sin t\left(1 + \frac{1}{2t}\sin 2t + ..\right) \qquad (18)\\ x_o &= -\dot{p}_o \end{aligned}$$

The second degree part in eq.17 describes the behaviour of H.F.phase and energy of geometrically centered particles. The third degree part gives one resonant coupling term between radial oscillations, H.F.phase and energy :

$$-\frac{\pi_x y^2}{6\rho t^{3/2}}\cos t \qquad (19)$$

This resonant coupling term drives the orbit centre along the acceleration gap and causes an energy decrease per turn in the particle energy proportional to y^2 (y is the orbit displacement at right angles to the gap). Together with a change in energy also the H.F.phase varies slightly.

3.2 Fourfold symmetric field and a two-dee system

A cyclotron with a four fold symmetric magnetic field and a two-dee system is schematically drawn in fig.4:

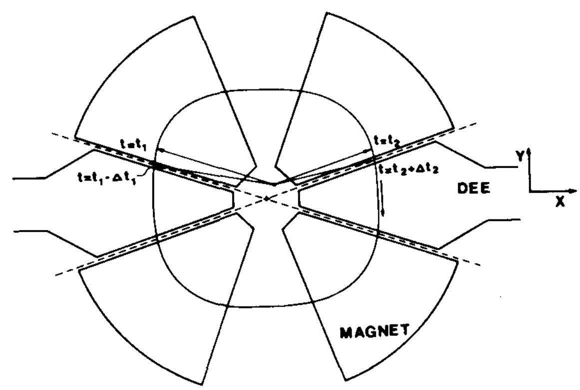

fig.4: *A fourfold symmetric magnetic field and a two-dee system.*

The treatment for this case is similar to the treatment in section 3.1, though more complicated. Because of the complexity of the formulas and the limited space in this paper we will only discuss some results.

The potential function is again split into two parts. One part is a pure time dependent function and can be omitted, the other part is a function (F) of all variables and is represented in a Fourier series. In fig.5 $He(x_o,p_{xo})$, $He(\xi+p_y+x_o, y+\pi_x+p_{xo})$ and $F=He(\xi+p_y+x_o, y+\pi_x+p_{xo})-He(x_o,p_{xo})$ are given for the push-push mode of operation.

Now one can find the Hamilton function as a power expansion of the variables. The quantities x_o, p_{xo}, y_o and p_{yo} which arise after transforming the Hamilton function such that the first degree parts disappear, determine the central accelerated orbit. The most important terms in these quantities are given in eq.20.

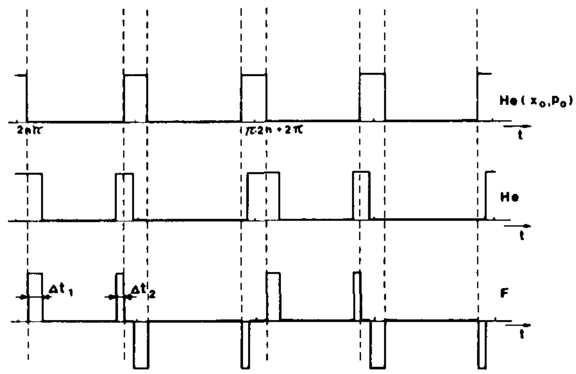

fig.5: *A representation of the functions $He(x_o,p_o)$, $He(x_o+\xi+p_y,p_o+\pi_x+y)$ and $F(\xi+p_y,\pi_x+y,t)$ as a function of time with constant argument π_x+y and $\xi+p_y$.*

$$x_o = \rho t^{\frac{1}{2}}\cos t \qquad \rho^2 = \frac{4\sin mt_o}{\pi} \qquad (20)$$

$$p_{xo} = -\rho t^{\frac{1}{2}}\sin t$$

$$y_o = \frac{f_n}{n^2-1}\, p_o - \frac{1}{\pi}\,\frac{4\sin^2\alpha\sin m\alpha}{\rho t^{\frac{1}{2}}}\cos t$$

$$p_{yo} = \frac{f_n}{n^2-1}\, x_o - \frac{1}{\pi}\,\frac{4\cos^2\alpha\sin m\alpha}{\rho t^{\frac{1}{2}}}\sin t$$

f_n represents the amplitude of the n^{th} harmonic field modulation, α equals half of the angle between the two accelerationgaps and m is the harmonic number of the acceleration frequency.

After the transformation yielding the central orbit coordinates given in eq. 20 a new Hamilton function arises in which coupling between the two, motions (x,p_x) and (y,p_y) occurs in the third degree terms. At first sight the analytical calculations shows that the coefficients of the resonant coupling terms are proportional to the square of the harmonic number of the acceleration voltage. Further extra terms (due to the acceleration mechanism) in the quadratic part change slightly the relation between H.F.phase and particle energy and also the value of the revolution frequency of the orbit centre (ν_r-1).
A complete treatment of this orbit theory will be published.

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