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SPACE CHARGE EFFECTS AND CURRENT LIMITATIONS IN CYCLOTRONS*

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Abstract

The effects of the electric fields generated by the ion beam in a cyclotron are investigated theoretically on the assumption that the beam consists of well defined bursts of ions moving on separated orbits. The theoretical analysis is carried out in two steps by first calculating the field acting on an ion at the surface of an individual burst due to the charge contained in the burst itself and then adding the contributions of charges on neighboring orbits. The electric field is found to be a function of the burst's radial position, inversely proportional to the velocity (or radius) on the first turns and then rapidly leveling off at intermediate radii. The influence of various parameters (dimensions of the bursts, energy gain per turn, etc.), the changes in the betatron frequencies and current limitations due to the space charge forces are discussed and illustrated in a numerical example.

1. Introduction

The repulsive action of the electric field due to the charge of the ions is one of the major problems in achieving higher phase-space densities of the ion beam in cyclotrons as well as in other accelerators. These space charge effects are, in general, not amenable to rigorous theoretical treatment. At the same time they limit the applicability of single-particle dynamics on which designers usually rely, and some analysis of the influence and magnitude of these effects is therefore of great importance.

Past estimates of space charge effects in cyclotrons and synchrocyclotrons assumed a continuous radial distribution of the circulating ion beam and approximated this beam by a flat, infinite sheet of uniform charge density which results in a constant electric field perpendicular to the plane of motion[1-3].

These approximations, however, are only valid for intermediate radii; at the "edges", i.e., in the center and near extraction the situation is different. Furthermore, the assumption of a continuous radial current distribution does not apply when individual turns are well separated as is the case in the central region of most cyclotrons and to an even greater extent in the proposed separated orbit cyclotron now under study at Oak Ridge National Laboratory[4] and Chalk River.

This paper deals with a theoretical study of the space-charge problem which is based on the assumption that the ion beam in the cyclotron consists of individual bursts of finite dimensions moving on separated orbits. The electric fields produced on the surface of a given burst due to the charge contained in the burst itself as well as the contributions of charges on neighboring turns are calculated in approximate form as a function of the orbit radius which defines the burst's position in the circulating beam. Formulas for the changes in the radial and vertical betatron frequencies and for the current limitations due to the space charge forces are presented. The influence of the various parameters (current, transverse and longitudinal dimensions of the bursts, energy gain per turn, total number of turns, etc.) is analyzed and some numerical results are presented.

2. General Considerations

For a general analysis of the problem let us consider the history of an individual "fish" of ions on the spiral-shaped path through the cyclotron from the first orbit to the extraction radius. Fig. 1 shows a schematic view of the situation in the early part of this acceleration process. For simplicity the "fish" is assumed to have a rectangular cross section of radial width 2x and vertical height 2z and to stretch along the circumference of the orbit radius with an azimuthal extent of $\Delta\phi$. (Deformations due to the difference in energy gain, different electric focusing and other effects are neglected). At the end of the first revolution the ion burst is at a radius R_1 which is determined by the voltage gain per turn, V_1. Other ion bunches are at radii $R_n = R_1\sqrt{n}$ where n is the number of turns.

For an evaluation of the space charge effects we can distinguish between the central region where in many cases the turn separation, ΔR, is large compared to the intrinsic width, 2x, of the ion beam and the remaining region where 2x is comparable to ΔR. On the initial orbits, an ion will predominantly be affected by the charge contained in the particular burst to which it belongs whereas at larger radii the influence of the ion charges on neighboring turns becomes increasingly important and eventually will be the dominant factor. As the extraction radius is approached this "proximity" effect will be reduced again and disappear completely when the beam has been extracted. (The special situation arising at larger radii when the turn separation becomes smaller than the radial width of

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the ion bunches and neighboring bursts do overlap will not be considered here).

The electric potential which a given ion experiences is thus the result of the superposition of the field generated by the burst to which it belongs and the contributions of all the charges on neighboring turns, V_n. To these two terms one has to add the influence of the image fields, V_i, resulting from the presence of the dees and other conducting electrodes. The potential at a point $P(r, \theta, z)$ on or near the surface of a given burst is then:

$$V(r,\theta,z) = \frac{1}{4\pi\epsilon_o} \int \frac{\rho d\tau}{s} + V_n(r,\theta,z) + V_i(r,\theta,z) \quad (1)$$

where the integral in the first term is to be taken over the volume of the burst; ρ is the charge density and s the distance between $P(r,\theta,z)$ and a point $P'(\xi,\varphi,\zeta)$ in the burst, i.e.

$$s^2 = (z-\zeta)^2 + r^2 + \xi^2 - 2r\xi \cos(\theta-\varphi).$$

None of the three terms in eq.(1) can be solved in elementary, closed form even with the idealized geometry of Fig. 1. We, therefore, will try approximate solutions which allow simple interpretation and evaluation of the various parameters. First, one can argue by reasons of symmetry, that the image charge term will be negligable when the beam height is small compared to the internal dee spacing, 2h. (In neglecting this term, however, one has to keep in mind that it may become very important when the beam blows up and fills the available space completely.) The remaining two terms will be treated as follows: (1) We restrict ourselves to calculating the field for the envelope ions at points $P_1(R,0,z)$ and $P_2(R+x,0,0)$ of the burst, where R is the orbit radius, 2x the radial, 2z the vertical extension of the burst. (2) In evaluating the first term, we assume that the azimuthal extension, $R\Delta\phi$, of the burst is large compared to the transverse dimensions but smaller than R so that $\cos \frac{\Delta\phi}{2}$ can be replaced by $1 - \frac{\Delta\phi^2}{4}$.
(3) To calculate the influence of other turns the ion bunches on these turns will be approximated by line charges of finite length $R\Delta\phi$.

Before this program is carried out in the next chapters, we will discuss briefly what implications the space charge force will have on the beam dynamics. The vertical electric field component $E_z = - \frac{\partial V}{\partial z}$ acting on the envelope ion at point P_1, i.e., at distance z from the median plane, results in a force qE_z which is opposed to the external focusing force. In the smooth approximation form of the linear betatron oscillations this can be expressed as:

$$m \frac{d^2z}{dt^2} + m\omega^2 \nu_z^2 z - qE_z = 0 \quad (2)$$

or

$$\frac{d^2z}{dt^2} + \omega^2 \left(\nu_z^2 - \frac{qE_z}{m\omega^2 z}\right) z = 0 \quad (3)$$

The net effect of the space charge force is then a reduction of the vertical betatron frequency, ν_z, the new effective frequency being given by the formula:

$$\nu_z^{*2} = \nu_z^2 - \frac{qE_z}{m\omega^2 z} \quad (4)$$

The relative change is:

$$\frac{\Delta\nu z}{\nu_z} = \left[1 - \frac{qE_z}{m\omega^2 \nu_z^2 z}\right]^{1/2} - 1 \quad (5)$$

which for small changes can be approximated by:

$$\frac{\Delta\nu z}{\nu_z} = - \frac{1}{2} \frac{qE_z}{m\omega^2 \nu_z^2 z} \quad (6)$$

The space charge limit is reached when $\frac{\Delta\nu z}{\nu_z} = -1$ or $\nu_z^* = 0$. This limit can be looked upon in two ways: (a) it determines the current density, i.e., the maximum current which for given ν_z can be contained in a beam of height z and thus represents a limit to the achievable vertical beam quality (measured in mA/cm rad); (b) ultimately, it limits the total current that can be accelerated in a cyclotron with dee height 2h and vertical betatron frequency ν_z. These limits are reached at the radius where $\nu_z{}^z$ has its minimum value, or better where $\frac{E_z}{\nu_z^2 z}$ is a maximum, and this usually occurs in the center of the cyclotron. If the current emerging from the ion source exceeds the space charge limit at this point, as it does in many cases, it will immediately start to spread vertically until at some larger radius the ν_z value is big enough to balance the space charge force; if the imitial current exceeds the space charge limit of the cyclotron ions will hit the dees and thus be removed from the beam until the external focusing force can contain the surviving current, as was discussed by Mackenzie[3].

The radial betatron frequency is also affected by the space charge forces. However, the situation is here less critical than in the case of the vertical motion since ν_r is generally greater than ν_z and since the radial field components are cancelled at intermediate radii. (The fields generated by the bursts inside and outside a given intermediate radius have opposite directions and therefore tend to cancel whereas the superposition is additive in the vertical direction). Where the

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radial field components do not vanish, as in the central region and on the outer radii, formulas (2) to (6) also obtain for the radial motion except that E_z, z, and ν_z are to be replaced by E_r, x, and ν_r.

3. Electric Field of a Single Ion Burst.

With the assumption of constant charge density ρ, restricting ourselves to the case $\theta = 0$, and using $\cos\varphi = 1 - \varphi^2/2$ we can write for the first term in Eq. (1):

$$V(r,0,z) = \frac{\rho}{4\pi\epsilon_o} \int_{R-x}^{R+x} \int_{-\frac{\Delta\phi}{2}}^{\frac{\Delta\phi}{2}} \int_{-z}^{z} \frac{\xi \, d\xi \, d\varphi \, d\zeta}{\sqrt{(z-\zeta)^2+(r-\xi)^2+r\xi\varphi^2}} \tag{7}$$

The coordinates of the envelope particles to which the calculation is restricted are $P_1(R,0,z)$ and $P_2(R+x,0,0)$. Since $x \ll R$ we can replace $r\xi$ by R^2. Integration of (7) under these conditions yields for the vertical component of the electric field at P_1:

$$E_z = -\frac{\partial V}{\partial z} = \frac{2x\rho}{4\pi\epsilon_o} G_z \tag{8}$$

and for the radial field component at P_2:

$$E_r = -\frac{\partial V}{\partial r} = \frac{2z\rho}{4\pi\epsilon_o} G_r \tag{9}$$

The constants G_z and G_r depend on the ratio of width to height of the bursts, $x/z = \alpha$, according to

$$G_z = \log\left(1 + \frac{4}{\alpha^2}\right) + \frac{4}{\alpha} \tan^{-1}\frac{\alpha}{2} \tag{10}$$

and

$$G_r = \log\left(1 + 4\alpha^2\right) + 4\alpha \tan^{-1}\frac{1}{2\alpha} \tag{11}$$

Fig. 2 shows G_z, G_r, and the ratio of the two field components, E_z/E_r, as a function of α. It is seen in this figure that for a large range of α values the ratio E_z/E_r is approximately unity.

The charge density ρ is proportional to the current and inversely proportional to the velocity v of the ions according to

$$\rho = \frac{I_o}{4xzv} = \frac{2\pi I}{\Delta\phi 4xzv} \tag{12}$$

I_o is the peak current, $I = \frac{\Delta\phi}{2\pi} I_o$ the average current, and 4xz the transverse cross section of the ion burst. Substituting (12) into (8) and (9) yields:

$$E_z = \frac{I_o G_z}{8\pi\epsilon_o vz} = \frac{IG_z}{4\epsilon_o \Delta\phi vz} \tag{13}$$

$$E_r = \frac{I_o G_r}{8\pi\epsilon_o vx} = \frac{IG_r}{4\epsilon_o \Delta\phi vx} \tag{14}$$

The field strength is thus seen to be equivalent to that of a line charge multiplied by a factor of $G_z/4$, or $G_r/4$. It is proportional to the current and inversely proportional to the velocity v and the distance z, or x, from the center of the burst.

The velocity v in turn is proportional to the cyclotron frequency ω and the orbit radius R:

$$v = R\omega = R_1\sqrt{n}\,\omega \tag{15}$$

The radius R_1 after the first revolution is related to the voltage gain per turn, V_1, and the magnetic field in the center, B_o:

$$R_1 = \frac{1}{B_o}\sqrt{\frac{2m}{q} V_1} \tag{16}$$

Note that ω is assumed to be constant (isochronous magnetic field) and relativistic effects are neglected. According to Eqs. (13) and (14) the electric field and the resulting force due to the space charge of a single ion burst have their maximum values at the ion source and decrease with increasing velocity.

Rapid initial acceleration of the ions to high velocities by applying large dee voltages or external injection with high kinetic energy will therefore reduce the defocusing effects of the space charge forces in the center of the cyclotron.

4. Influence of Ion Bursts on Neighboring Orbits

To evaluate the second term in Eq. (1) we approximate the neighboring bursts by finite line charges. If we choose a rectangular coordinate system y,s where the line charge is located along the y axis extending from $y = -L/2$ to $y = +L/2$ the transverse field strength at distance s from the y axis is

$$E_s = \frac{I_o}{4\pi\epsilon_o v} \frac{1}{s}\left(\frac{y+L}{r_1} - \frac{y-L}{r_2}\right) \tag{17}$$

where r_1 and r_2 are the distances from the two end points of the line charge. We restrict ourselves to the case $y = 0$ and obtain

$$E_s = \frac{I_o}{2\pi\epsilon_o vs} \frac{1}{\sqrt{1 - s^2/L^2}} \tag{18}$$

At small distances ($s \ll L$) this expression approximates the field of an infinite line charge whereas at large distances ($s \gg L$) it resembles the field of a point charge with a $1/s^2$ dependence. Let us now consider an ion burst at a given radius $R_n = R_1\sqrt{n}$. The electric field strength produced at

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the envelope point $P_1(R_n,z)$ by the charge of some other burst located at radius R_ν, is given by Eq. (18) with

$$s = s_\nu = \sqrt{z^2 + (R_\nu - R_n)^2} \qquad (19)$$

The vertical motion is effected by the z-component of this field strength which is

$$E_{z\nu} = E_{s\nu} \frac{z}{s_\nu} \qquad (20)$$

The total vertical field strength is obtained by summation of the contributions of all neighboring turns from $\nu = 1$ to $\nu = n-1$ and from $\nu = n+1$ to $\nu = n_t$ where n_t is the total number of turns in the cyclotron:

$$\sum_{\substack{\nu=1\\ \nu\neq n}}^{n_t} E_{z\nu} = \sum_{\substack{\nu=1\\ \nu\neq n}}^{n_t} E_{s\nu}\frac{z}{s_\nu} = \frac{I_o}{2\pi\epsilon_o}\sum_{\substack{\nu=1\\ \nu\neq n}}^{n_t} \frac{1}{v_\nu s_\nu\sqrt{1+(s_\nu/L_\nu)^2}}\,\frac{z}{s_\nu} \qquad (21)$$

with $v_\nu = \omega R_1\sqrt{\nu}$, $R_\nu = R_1\sqrt{\nu}$, and $L_\nu = R_\nu\frac{\Delta\phi}{2} = R_1\sqrt{\nu}\,\frac{\Delta\phi}{2}$

we obtain

$$\sum_{\substack{\nu=1\\ \nu\neq n}}^{n_t} E_{z\nu} = \frac{I_o z/R_1}{2\pi\epsilon_o\omega R_1^2}\sum_{\substack{\nu=1\\ \nu\neq n}}^{n_t}\left(\sqrt{\nu}\left[(z/R_1)^2+(\sqrt{\nu}-\sqrt{n})^2\right]\right)^{-1} \times$$

$$\times\left(1+\frac{4}{\nu\Delta\phi^2}\left[(z/R_1)^2+(\sqrt{\nu}-\sqrt{n})^2\right]\right)^{-1/2} \qquad (22)$$

The vertical field strength due to charges on neighboring turns is thus a function of n, i.e., the position of the ion under consideration, with R_1, the beam height, z, the azimuthal length, $\Delta\phi$, and the total numer of turns, n_t, as parameters.

The total field strength acting on the vertical envelope particle of a given burst is obtained by adding the field produced by the burst itself, Eq. (13), and the contribution of other turns, Eq. (22):

$$E_z = \frac{I_o}{\epsilon_o\omega R_1^2}F(n) = \frac{I}{\epsilon_o\omega\left(\frac{\Delta\phi}{2\pi}\right)R_1^2}F(n) \qquad (23)$$

where

$$F(n) = \frac{G_z}{8\pi(z/R_1)\sqrt{n}} + \frac{z/R_1}{2\pi}\sum_{\substack{\nu=1\\ \nu\neq n}}^{n_t}\left(\sqrt{\nu}\left[(z/R_1)^2+(\sqrt{\nu}-\sqrt{n})^2\right]\right)^{-1}\times$$

$$\times\left(1+\frac{4}{\nu\Delta\phi^2}\left[(z/R_1)^2+(\sqrt{\nu}-\sqrt{n})^2\right]\right)^{-1/2} \qquad (24)$$

The factor $I_o/\epsilon_o\omega R^2$ in Eq. (23) represents the electric field due to an infinite plane of uniform charge distribution which we will denote with E_{zc}. Since $v_1R_1 = v_1^2/\omega = 2qV_1/m\omega$, we can write

$$E_{zc} = \frac{I_o}{\epsilon_o\omega R_1^2} = \frac{I_o m\omega}{2\epsilon_o qV_1} = \frac{Im\omega}{2\epsilon_o\frac{\Delta\phi}{2\pi}qV_1} \qquad (25)$$

This is the formula for the space charge field used in the literature.[1-3] The function F(n) will be discussed in the next chapter.

The change of the vertical frequency due to the space charge field is obtained by substituting Eq. (23) into Eq. (5) using (25) which yields:

$$\frac{\Delta\nu_z}{\nu_z} = \left[1 - \frac{IF(n)}{2\epsilon_o\frac{\Delta\phi}{2\pi}\,\omega zV_1\nu_z^2}\right]^{1/2} - 1 \qquad (26)$$

The maximum current which can be contained in a beam of height z is

$$I_{lim} = 2\epsilon_o\left(\frac{\Delta\phi}{2\pi}\right)\omega zV_1\nu_z^2\,\frac{1}{F(n)} \qquad (27)$$

A similar procedure can be applied to the calculation of the radial field component, E_r, acting on an envelope particle in the median plane. If the ion burst considered is at radius $R_n = R_1\sqrt{n}$ and the envelope particle is at point $R_n + x$ the distance to some other burst at radius R_ν becomes

$$s_\nu = R_n + x - R_\nu = R_1(\sqrt{n}-\sqrt{\nu}+\frac{x}{R_1}) \qquad (28)$$

and the summation over all turns inside and outside of turn n yields

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$$\sum_{\substack{\nu=1\\ \nu\neq n}}^{n_t} E_{r\nu} = \frac{I_o}{2\pi\epsilon_o \omega R_1^2} \sum_{\substack{\nu=1\\ \nu\neq n}}^{n_t} \left(\sqrt{\nu}\left[\sqrt{n}-\sqrt{\nu} + x/R_1\right]\right)^{-1} \times$$

$$\times \left(1 + \frac{4}{\nu\Delta\phi^2}\left[\sqrt{n}-\sqrt{\nu} + x/R_1\right]^2\right)^{-1/2} \qquad (29)$$

Note that for turns outside $n(\nu>n)$ the terms in the sum have negative signs and hence will be subtracted from the terms for $\nu<n$. The total field strength is obtained by adding Eqs. (14) and (29):

$$E_r = \frac{I_o}{\epsilon_o \omega R_1^2} G(n) = \frac{I}{\epsilon_o \left(\frac{\Delta\phi}{2\pi}\right) \omega R_1^2} G(n) \qquad (30)$$

where

$$G(n) = \frac{G_r}{8\pi(x/R_1)\sqrt{n}} + \frac{1}{2\pi}\sum_{\substack{\nu=1\\ \nu\neq n}}^{n_t} \left(\sqrt{\nu}(\sqrt{n}-\sqrt{\nu} + x/R_1)\right)^{-1} \times$$

$$\times \left(1 + \frac{4}{\nu\Delta\phi^2}(\sqrt{n}-\sqrt{\nu} + \frac{x}{R_1})^2\right)^{-1/2} \qquad (31)$$

The change in radial frequency due to this space charge field is

$$\frac{\Delta\nu_r}{\nu_r} = \left[1 - \frac{IG(n)}{2\epsilon_o \left(\frac{\Delta\phi}{2\pi}\right)\omega V_1 x \nu_r^2}\right]^{1/2} - 1 \qquad (32)$$

It should be noted that in order to avoid a singularity in the function $G(n)$ Eq. (31) can only be used for turn numbers n where x is smaller than the turn separation. A more stringent requirement is that the total radial width, 2x, of the beam should always be smaller than the turn separation. This limits the orbit number for which $G(n)$ can be calculated by means of Eq. (31) to values where $\sqrt{n+1} - \sqrt{n} \geq 2x/R_1$ or

$$n \leq \frac{1}{16}\left(R_1/x\right)^2 \qquad (33)$$

5. Analysis of the Theoretical Results

The function $F(n)$ in Eq. (24) which through the relation $r = R_n = R_1\sqrt{n}$, also constitutes a function of radius, r, measures the ratio of the field E_z to the field which would be produced by a continuous infinite beam with uniform charge density. One would therefore expect that this function approaches unity for intermediate turns provided that the total number of turns, n_t, is large. To show this we approximate the summation in Eq. (24) by an integral neglecting the correction factor

$$\left[1 + (s_\nu/L_\nu)^2\right]^{-1/2}$$

Now

$$\int_1^{n_t} \frac{d\nu}{\sqrt{\nu}\left[(z/R_1)^2 + (\sqrt{\nu}-\sqrt{n})^2\right]} =$$

$$= \frac{2}{z/R_1}\left[\tan^{-1}\frac{\sqrt{n_t}-\sqrt{n}}{z/R_1} + \tan^{-1}\frac{\sqrt{n}-1}{z/R_1}\right] \qquad (34)$$

Eq. (24) takes then the form:

$$F(n) = \frac{G_z}{8\pi(z/R_1)\sqrt{n}} + \frac{1}{\pi}\left[\tan^{-1}\frac{\sqrt{n_t}-\sqrt{n}}{z/R_1} + \tan^{-1}\frac{\sqrt{n}-1}{z/R_1}\right] -$$

$$- \frac{1}{2\pi}\frac{1}{(z/R_1)\sqrt{n}} \qquad (35)$$

This formula is good for not too small values of n. If n is of the order $n_t/2$ and n_t is very large the two arctan functions each approach maximum values close to $\pi/2$. Since both the first and last term in Eq. (35) go to zero for large n the function $F(n)$ does indeed approximate unity in this limit as expected.

For small values of n, i.e., on the central orbits, the influence of neighboring turns diminishes and the space charge field is mainly determined by the first term in Eqs. (24) and (35). The extent to which this term dominates depends primarily on the turn separation in the center, which is determined by the radius R_1, and to some degree on the total number of turns in the cyclotron.

To obtain a quantitative picture of the situation, let us consider the hypothetical case of a 40-MeV deuteron cyclotron with fixed magnetic field and therefore fixed extraction radius. The width and height of the beam are assumed to be constant, with x = 1 mm, z = 5 mm, and hence $G_z = 6.61$. The central magnetic field, B_o, is assumed to be 12.9kG which gives a value of $\omega = 6.19 \times 10^7\ s^{-1}$. With these parameters the function $F(n)$ was calculated for two cases of voltage gain per turn, (a) V_1 = 100kV, and (b) V_1 = 400kV; the corresponding values

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for the first orbit radius and the total number of turns are $R_1 = 5$ cm, $n_t = 400$ for case (a), and $R_1 = 10$ cm, $n_t = 100$ for case (b). The angular width of the beam was assumed to be $\Delta\phi = .5$ rad.

The results of the calculations with these conditions are plotted in Fig. 3 which shows F(n) as a function of radius R and $\sqrt{n}$ for the two cases. The broken lines in this figure represent the contributions of the single-burst term

$$F_1(n) = G_z/(8\pi\sqrt{n}z/R_1)$$

in Eq. (24). The plots exhibit the expected behavior. In both cases $F_1(n)$ dominates at small radii and decreases rapidly with radius. The influence of neighboring turns rises very quickly in case (a) due to the low energy gain and small turn separation and after some 30 or 40 turns the function F(n) levels off at a value slightly above 1; near the final radius F(n) decreases again as expected. In case (b) the influence of neighboring bursts is less pronounced and increases less rapidly with radius than in case (a) contributing about 5 per cent on the fourth turn and a maximum of a little over 50 per cent at the outer radii before it decreases again. Note also that F(n) in this case is always greater than 1 at intermediate radii and thus does not approach the situation of a uniform charge distribution. The important point in Fig. 3 is the steep increase of F(n) at small radii which indicates that the electric field due to the space charge is much stronger than would be the case with a uniform charge distribution. The reason for this difference lies, of course, in the fact that the charge contained in one orbit is concentrated in a relatively small, distinct burst rather than being distributed uniformly over the space Δ R between successive turns. The deviation from the uniform-field case is roughly proportional to R_1 as was pointed out earlier.

The net effect of increasing the dee voltage, and thus of R_1, is twofold: (a) it reduces the electric field due to the space charge on the early turns roughly by a factor $1/R_1$ or $1/\sqrt{V_1}$ and on intermediate turns by $1/R_1^2$ or $1/V_1$, and (b) it moves the injection point to a larger radius where the magnetic focusing (which usually increases with radius in the center) is better. When F(n) is known one can calculate the change in the vertical betatron frequency, ν_z, that a given current produces or one can determine the maximum current which, for given dee height, z = h, and given radial dependence of ν_z, can be accelerated in a cyclotron. As an illustration we shall calculate the values of ν_z necessary to just compensate for the space charge of a given current in the case of the deuteron cyclotron discussed previously. Using the assumed values for $\Delta\phi$, ω, and z, we obtain from Eq. (27):

$$\nu_z^2 = 2.29 \frac{I[mA]F(n)}{V_1[kV]} \qquad (36)$$

In case (a) where $V_1 = 100$kV we find at the end of the first turn (n=1) in Fig. 3a the value F(n) = 2.69, which yields $\nu_z^2 = 6.17 \times 10^{-2}$ I[mA]. Thus, for a current of 1 mA one needs a value of $\nu_z = .248$ to compensate the space charge force. Such a high ν_z value is generally difficult to achieve at a radius of 5 cm.

In case (b), on the other hand, $V_1 = 400$kV and F(n) = 5.29 for n = 1 leading to $\nu_z^2 = 3.03 \times 10^{-2}$ I[mA] which is about half of the previous value. A current of 1 mA requires a balancing ν_z value in this case of about .174 which is still relatively high but easier to achieve as the first orbit radius in this case is 10 cm.

This numerical example demonstrates the magnitude of the space charge effects at the early orbits in a cyclotron. Since the magnetic betatron frequency, ν_z, generally rises from zero at the center to values of about .2 at radii between 10 cm and 20 cm (depending on the height of the magnet gap and other factors) it is clear that relatively low beam currents can easily lead to defocusing and possible losses to the dees. In any case, the effective vertical frequency may be substantially less than the values obtained from the single-particle theory. This applies, of course, also to the radial frequency, ν_r, at small radii. With the values of our previous example and taking only the dominant first term in Eq. (31) we get G(n)= 2.2 for n=1 in case (a) and G(n) = 4.4 for n=1 in case (b). The relative change in ν_r can be calculated from Eq.(32). Assuming a value of ν_r=1.0 one finds that a current of 1 mA results in a fractional change of the radial frequency of -14% in case (a) and of -7% in case (b). This is an appreciable reduction of the radial frequency which implies that the ν_r=1 resonance in sector focused cyclotrons is shifted to larger radii when relatively high currents are accelerated.

6. Conclusion

The preceding study of the space charge effects in cyclotrons is clearly a simplification of a very involved physical problem which does not allow a rigorous theoretical treatment. As was pointed out earlier, the individual ion bursts, under the influence of the external magnetic and electric fields and the space charge forces, undergo many changes of their shape associated with fluctuations in charge density. The assumptions made in the theory are therefore approximations of the average behavior of the ion beam and of the average magnitude of the space charge effects in a similar sense that the linear betatron frequencies are an approximate measure of the average focusing forces in a sector-focused cyclotron.

Apart from these restrictions the formulas derived allow approximate calculations of the electric fields generated by the circulating ion charges on the surface of the central part of a burst and permit quantitative estimates of the resulting changes of the betatron frequencies or of the total current that can be accelerated in a beam of given dimensions.

References

1). H.G. Blosser and M.M. Gordon, Nucl. Instr. and Meth. 13, 101 (1961).

2). A.A. Glazov et al., Proc. Intern. Conf. on High Energy Accelerators, Dubna, p. 786, (1963).

3). K.R. Mackenzie, Nucl. Instr. and Meth. 31, 139, (1964).

4). F.M. Russel, Nucl. Instr. and Meth. 23, 229, (1963).

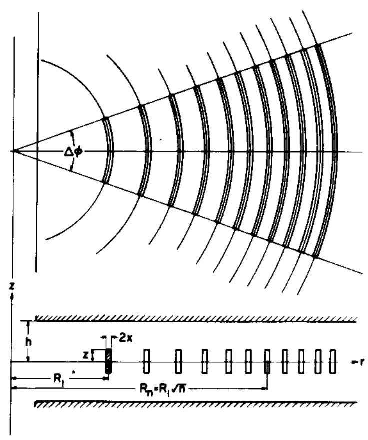

Fig. 1. Schematic view of the circulating ion bursts on the first turns in the center of a cyclotron.

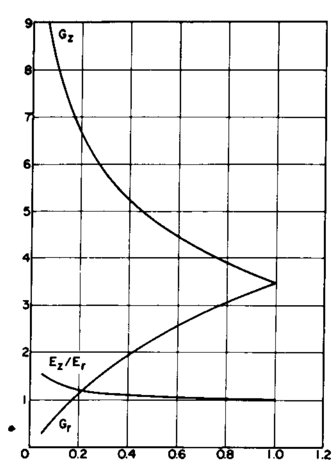

Fig. 2. Geometry factors G_z and G_r and field strength ratio E_z/E_r as functions of $a = x/z$.

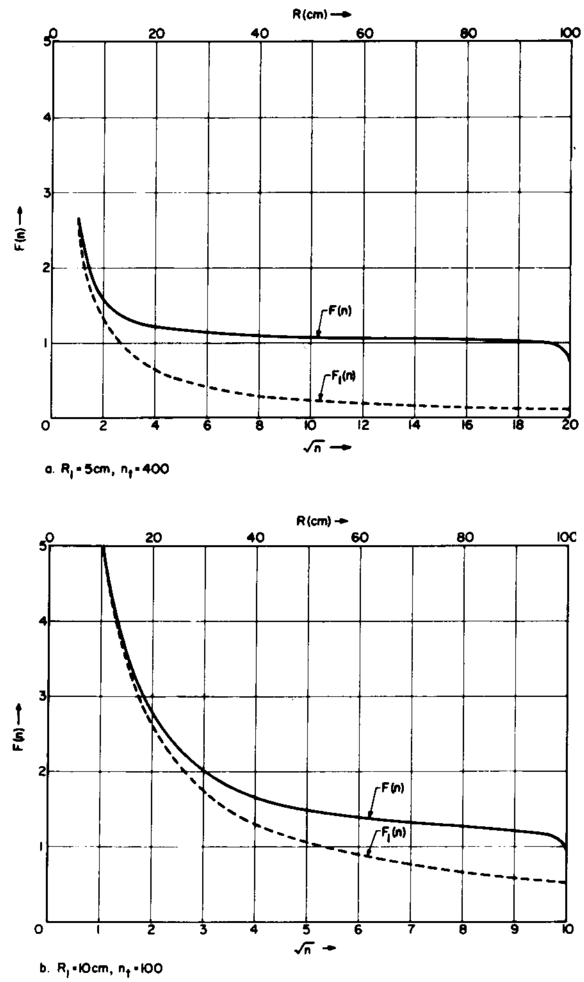

Fig. 3. Radial dependence of the function F(n) for two different orbit geometries. The broken line represents the relative field strength due to a single burst.

DISCUSSION

KOLSTAD: What is the effect of ignoring the image-charge term?

REISER: This depends on the height of the beam z, and the dee aperture h. I have not made the calculations because they are very involved. You have to consider the image charges generated by all the bursts that are simultaneously rotating in the cyclotron, and you get summation of a large number of terms. But it can be done. For example, I just considered one single burst and took into account the first few images induced in the conducting planes. I found the effect to be only a few percent in this case. But I should emphasize that whenever the beams blows up and hits the dees, the image-charge forces start to become important. I calculated in one case that it may very easily go up to 30%, then, of course, my assumptions do no longer give the real picture of the situation. Also, I neglected the magnetic field, for simplicity, and just considered non-relativistic situations. But, adding magnetic field and relativistic effects is, of course, straightforward.

POWELL: I should like to ask you a question about space charge in the median plane. I am thinking of some work by Welton reported at the Sea Island Conference. He pointed out that the leading edge of a bunch would gain energy, and the lagging edge would lose energy as a result of these space-charge forces. For a single-turn cyclotron, with a very narrow phase width, what energy spread would you expect from this sort of effect?

REISER: This effect surely exists; it is not very easy to calculate, but it can be done. (I have done something in that direction, but have not yet arrived at clear results.) I neglected here the electric field of the azimuthal edges of the bursts and considered only the central part of the burst.

BLOSSER: I would like to point out that Welton also showed how to get rid of the energy spread, by shifting the groups off from the peak to the side of the rf wave.