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CALCULATING THE DISTURBED MOTION OF PARTICLES IN CYCLIC ACCELERATORS

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Abstract

A few examples demonstrate an elementary method of calculating the closed orbit in a cyclical accelerator when perturbative forces are present. The free oscillations can be found in a similar way, the criterion for stability being the same as for the strong focusing synchrotron. A steadily increasing extraction force will, however, destroy the condition necessary for resonance.

In most cyclic and spiral particle accelerators, the orbit is never a smooth circular or spiral curve as determined by azimuthally constant stabilizing forces but, in most cases, it is a complicated path deformed by various azimuthally distributed perturbative forces. In such cases, the periodic disturbing forces acting on the particles can be represented as Fourier-series of the azimuthal coordinate thus giving another Fourier-series as the solution for the orbital curve. Using a computer for the numerical calculations, this method is straightforward and easy, and even non-linear forces and couplings between radial and axial components can be considered.

The method has, however, the usual draw-back of most computer calculations: it destroys the direct mental connections between the acting force and the resulting disturbance of the orbit thus making it for instance difficult to imagine how special forces have to be applied in order to obtain certain deviations from the normal orbit.

To enlighten the relations between the applied forces and the resulting changes of the orbit, some direct calculations of special cases might therefore still be motivated. In the following calculations, we will restrict our investigation to the very simple case of linear forces acting in the orbital plane only, thus leaving out all considerations on axial forces and disturbances and also their coupling with the radial movement of the particles. Mass and energy of the particles will be taken as constant. As usual, the circular orbit is replaced by a linear motion, without of course omitting both the centrifugal force acting on the particles, and the periodic character of the perturbations.

If y is the deviation from the circular orbit with the radius R, $m\ddot{y}$ is the mass force produced by the deviation, and $m\omega^2(1-n)y$ the restraining stabilizing force resulting from the difference between the Lorentz-force of the guiding magnetic field B (with the exponent $n = -\frac{R}{B}\frac{dB}{dR}$) and the centrifugal force $\frac{mv^2}{R}$ of the circular motion. The perturbative force will be denoted by mP, it being linearly dependent on the deviation y thus: $mP = mP_0\ (1 + py)$.

The differential equation for the forces will then be:

$$m\ddot{y} + m\omega^2\ (1-n)y + mP_0\ (1+py) = 0 \qquad (1)$$

resp.

$$\ddot{y} + \omega^2\ \underbrace{\left[\left(1 - n + \frac{pP_0}{\omega^2}\right)\right]}_{k}\ y + P_0 = 0 \qquad (1a)$$

where ω is the revolution frequency (= angular velocity) of the particles, ωt = x the azimuthal angle and P_0 a periodic function of the azimuth. As usual, the movement of the particles can be described by a superposition of the stationary solution of the equation (1a) (closed orbit, orbit of equilibrium, enforced oscillation) and free oscillations y_f (betatron oscillations) which are solutions of the differential equation (1b):

$$y_f'' + \omega^2\left[\overbrace{(1-n) + P_0p/\omega^2}^{k}\right]y_f = 0 \qquad (1b)$$

Our first task will be to determine the closed orbit.

As a first example, we will consider a constant perturbation P_1 only acting over the azimuthal angle $2x_1$ as shown in Fig. 1. The solution of the differential equation (1a) is:

$$y = C\cos\sqrt{k}\cdot x - \frac{P_0}{\omega^2 k} \qquad (2)$$

with the frequency-coefficient

$$k = (1-n) + \frac{pP_0}{\omega^2} \qquad (2a)$$

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The constant C is determined by the initial conditions of the particle. As can be seen, the orbital curve is composed of two parts which are joined together at the point y_1 where the perturbation begins. Choosing $x = 0$, $y = y_{01}$ and $y' = 0$ for the initial values at the midpoint of the perturbation, this part of the orbit gives us:

$$y_1 = \left(y_{01} + \frac{P_1}{\omega^2 k_1}\right) \cos \sqrt{k_1} x_1 - \frac{P_1}{\omega^2 k_1} \qquad (3)$$

and

$$y_1' = -\left(y_{01} + \frac{P_1}{\omega^2 k_1}\right) \sqrt{k_1} \sin \sqrt{k_1} x_1 \qquad (3a)$$

Calculating the same values for the undisturbed part of the orbit ($x = x_2$, $k = k_2$) we get:

$$y_1 = y_{02} \cos \sqrt{k_2} x_2 \qquad (4)$$

and

$$y_1' = y_{02} \sqrt{k_2} \sin \sqrt{k_2} x_2 \qquad (4a)$$

Comparing the expression (3) with (4) and (3a) with (4a), the unknown amplitudes of the orbit are found to be:

$$y_{01} = \frac{P_1}{\omega^2 k_1}\left[\frac{\sin \sqrt{k_2} x_2}{\sqrt{\frac{k_1}{k_2}} \sin \sqrt{k_1} x_1 \cos \sqrt{k_2} x_2 + \cos \sqrt{k_1} x_1 \sin \sqrt{k_2} x_2}, - 1\right] \qquad (5)$$

and

$$y_{02} = -\frac{P_1}{\omega^2 k_1} \frac{\sin \sqrt{k_1} x_1}{\sin \sqrt{k_1} x_1 \cos \sqrt{k_2} x_2 + \sqrt{\frac{k_2}{k_1}} \cos \sqrt{k_1} x_1 \sin \sqrt{k_2} x_2} \qquad (5a)$$

As a numerical example we take mP_1 to be 1% of the centrifugal force $m\omega^2 R$, the azimuthal length of the perturbation $2x_1 = 60°$ and $k_1 = k_2 = 0.5$, and we then get the amplitudes:

$$\frac{y_{01}}{R} = +0.416\%$$

$$\frac{y_{02}}{R} = -0.91\%$$

with the mean value:

$$\frac{y_{01} + y_{02}}{2R} = -0.247\%$$

In formula (1) we have assumed the perturbative force as being positive when directed inwards from the circular undisturbed orbit (i.e. in the same direction as the Lorentz-force) and as seen from formula (5) and (5a), this will produce an outward directed peak at the center of the perturbation (y_{01} is positive) whereas the mean value will be inside (i.e. negative) the undisturbed orbit A-A.[1]

Fig. 1 clearly shows how the momentum of the particles moving outwards is counteracted by the inward directed positive perturbation force thus shifting the radial velocity of the particles from the outward to the inward direction.

We have here assumed the frequency-coefficients k_1 and k_2 to be positive and the closed orbit will thus be composed of two sinusoidal parts.

If the frequency-coefficient k_1 is zero or negative, the calculation is very similar; as a result, we now get parabolic and hyperbolic parts of the curve as indicated by the following formulae (6), (6a), (7), and (7a):

<u>$k_1 = 0$</u>

$$y_{01} = \frac{P_1 x_1}{\omega^2}\left[\frac{x_1}{2} - \frac{1}{\sqrt{k_2} \operatorname{tg} \sqrt{k_2} x_2}\right] \qquad (6)$$

$$y_{02} = -\frac{P_1 x_1}{\omega^2} \frac{1}{\sqrt{k_2} \sin \sqrt{k_2} x_2} \qquad (6a)$$

and:

<u>k_1 = negative</u>

$$y_{01} = \frac{P_1}{\omega^2 k_1}\left[\frac{\sin \sqrt{k_2} x_2}{\sqrt{\frac{-k_1}{k_2}} \sinh \sqrt{-k_1} x_1 \cos \sqrt{k_2} x_2 + \cosh \sqrt{-k_1} x_1 \sin \sqrt{k_2} x_2} - 1\right] \qquad (7)$$

$$y_{02} = \frac{-P_1}{\omega^2 k_1} \frac{\sinh \sqrt{-k_1} x_1}{\sinh \sqrt{-k_1} x_1 \cos \sqrt{k_2} x_2 - \sqrt{\frac{k_2}{-k_1}} \cosh \sqrt{-k_1} x_1 \sin \sqrt{k_2} x_2} \qquad (7a)$$

As can be seen from Fig. 2, such values for k_1 will not change the character of the closed orbit very much.

As a second example, we will explain the calculation of the stationary curve when two different constant perturbations are acting on the particles. In Fig. 3, we have shown two perturbations P_1 and P_2 of the same azimuthal extension x_1, 120° apart and with positive frequency-coefficients k_1 and k_2, whereas the undisturbed part of the circumference has the coefficient k_3.

Considering the first parts of the orbit (−210° to +110° in Fig. 3), we can calculate the amplitudes of the oscillations by means of the formulae (5) and (5a) to be:

$$y_{01} = \frac{P_1}{\omega^2 k_1}\left[\frac{\sin\sqrt{k_3}x_3}{\sqrt{\frac{k_1}{k_3}}\sin\sqrt{k_1}x_1\cos\sqrt{k_3}x_3 + \cos\sqrt{k_1}x_1\sin\sqrt{k_3}x_3} - 1\right] \quad (8a)$$

and

$$y_{03} = \frac{-P_1}{\omega^2 k_1}\ \frac{\sin\sqrt{k_1}x_1}{\sin\sqrt{k_1}x_1\cos\sqrt{k_3}x_3 + \sqrt{\frac{k_3}{k_1}}\cos\sqrt{k_1}x_1\sin\sqrt{k_3}x_3} \quad (8b)$$

Looking at the second part of the orbit (+110° to 250° in Fig. 3), we get:

$$y_{02} = \frac{P_2}{\omega^2 k_2}\left[\frac{\sin\sqrt{k_3}x_4}{\sqrt{\frac{k_2}{k_3}}\sin\sqrt{k_2}x_2\cos\sqrt{k_3}x_4 + \cos\sqrt{k_2}x_2\sin\sqrt{k_3}x_4} - 1\right] \quad (8c)$$

$$y_{03} = \frac{-P_2}{\omega^2 k_2}\ \frac{\sin\sqrt{k_2}x_2}{\sin\sqrt{k_2}x_2\cos\sqrt{k_3}x_4 + \sqrt{\frac{k_3}{k_2}}\cos\sqrt{k_2}x_2\sin\sqrt{k_3}x_4} \quad (8d)$$

Comparing the two expressions for the amplitude y_{03} and remembering that $x_4 = 180° - (x_1 + x_2 + x_3)$, we now get an equation for calculating the unknown azimuthal value x_3 and the amplitudes of the stationary curve can thus also easily be found.

Taking as a numerical example (from the development of a 100 MeV electron-synchrotron), we might for instance have: $P_1 = +8.1\%$, $P_2 = +10.4\%$ of $m\omega^2 R$, $k_1 = k_2 = k_3 = 0.5$, $x_1 = x_2 = 60°$ and calculate the amplitudes of the stationary orbit to be: $y_{01}/R = -6.1\%$, $y_{02}/R = -3.04\%$, and $y_{03}/R = -7.7\%$ with $x_3 = 40°$.

As can be seen from Fig. 3, the disturbing forces have, in this case, placed the resulting orbit far inside the undisturbed orbit with peaks directed in the opposite direction to the perturbating forces, i.e. outwards.

It is easy to extend this method so as to calculate the closed orbit for a plurality of disturbing constant forces.

As a last example, we shall show how to apply the perturbing forces when we want to create a special deviation, for instance a "bump," in the stationary orbit. As can be seen from Fig. 4, the "bump" we want has an azimuthal extension of $2(x_1 + x_2)$ leaving the rest of the orbit without any alteration. Again the deviation can be composed of two (in this case sinusoidal) parts, one created by the outward directed forces, and another part which changes the radial velocity of the particles thus caused by applying an inward directed force. Using equation (2) on the first part, we get the following values:

$$y_1 = \frac{P_1}{\omega^2 k_1}\left(\cos\sqrt{k_1}x_1 - 1\right) \quad (9)$$

and

$$y_1' = \frac{-P_1}{\omega^2\sqrt{k_1}}\sin\sqrt{k_1}x_1 \quad (9a)$$

for the point on the curve where the two parts meet. The second part of the curve will give us:

$$y_1 = \left(y_a + \frac{P_2}{\omega^2 k_2}\right)\cos\sqrt{k_2}x_2 - \frac{P_2}{\omega^2 k_2} \quad (10)$$

$$y_1' = +\left(y_a + \frac{P_2}{\omega^2 k_2}\right)\sqrt{k_2}\sin\sqrt{k_2}x_2 \quad (10a)$$

for the same point. The equations (9), (9a) and (10), (10a) allow us to calculate the forces P_1 and P_2 when the peak y_a of the deviation is given.

$$P_1 = -y_a\omega^2 k_1 \frac{\sin\sqrt{k_2}x_2}{\sqrt{\frac{k_1}{k_2}}\sin\sqrt{k_1}x_1\,(1-\cos\sqrt{k_2}x_2)+\sin\sqrt{k_2}x_2\,(1-\cos\sqrt{k_1}x_1)} \qquad (11)$$

$$P_2 = +y_a\omega^2 k_2 \frac{\sqrt{\frac{k_1}{k_2}}\sin\sqrt{k_1}x_1\,\cos\sqrt{k_2}x_2 - \sin\sqrt{k_2}x_2\,(1-\cos\sqrt{k_1}x_1)}{\sqrt{\frac{k_1}{k_2}}\sin\sqrt{k_1}x_1\,(1-\cos\sqrt{k_2}x_2)+\sin\sqrt{k_2}x_2\,(1-\cos\sqrt{k_1}x_1)} \qquad (11a)$$

As a numerical example, we choose: $k_1 = k_2 = 0.5$, $x_1 = 36°$, $x_2 = 18°$ and $y_a/R = 1\%$ and obtain for the forces $mP_1 = -3.50\%$, $mP_2 = +6.31\%$ of $m\omega^2 R$. The peak of the deviation will become higher and sharper when the inwardly directed force is made greater and is acting over a smaller azimuthal range.

We will now investigate the free oscillations of the particles around the closed orbit. The solutions of the differential equation (1b) will again give us a curve composed of sinusoidal or hyperbolic (resp. parabolic) parts in the same way as for the closed orbit. The circumference might have azimuthal parts with various values for the frequency-coefficient k and the free oscillation will then also contain various parts which have to be fitted together at the boundaries where the k-values change. We can for instance write the first part of the oscillations as follows:

$$y = A_1\cos\sqrt{k_1}x + B_1\sin\sqrt{k_1}x \qquad (12)$$

$$y' = -A_1\sqrt{k_1}\sin\sqrt{k_1}x + B_1\sqrt{k_1}\cos\sqrt{k_1}x \qquad (12a)$$

where A_1 and B_1 are constants determined by the initial conditions of the free oscillation. At the boundary where the frequency-coefficient changes to k_2, we consequently get:

$$y_1 = A_1\cos\sqrt{k_1}x_1 + B_1\sin\sqrt{k_1}x_1 = A_2 \qquad (13)$$

$$y_1' = -A_1\sqrt{k_1}\sin\sqrt{k_1}x_1 + B_1\sqrt{k_1}\cos\sqrt{k_1}x_1 = B_2\sqrt{k_2} \qquad (13a)$$

In this way, we can calculate the constants A_2 and B_2 for the next part of the orbit and, successively in the same way, the complete oscillation. If the frequency-coefficients are negative, the corresponding part of the curve will be hyperbolical. In this case, the possibility arises that the oscillations might become unstable with steadily increasing amplitudes resulting in an ejection of the particles.[2,3] Our case is exactly the same as for the strong focusing cyclic accelerators and from their theory, we can take the criterion for stability to be:

$$-1 < \left(\cos\sqrt{k_1}x_1\,\cos\sqrt{k_2}x_2 - \frac{k_1+k_2}{2\sqrt{k_1k_2}}\sin\sqrt{k_1}x_1\,\sin\sqrt{k_2}x_2\right) < +1 \qquad (14)$$

When the expression shown in brackets exceeds the limits +1 or −1, the oscillation will become unstable. Let us for instance take a perturbation acting over $x_1 = 60°$ with an outward decreasing force giving the negative frequency-coefficient k_1 with the frequency-coefficient for the rest of the circumference being $k_2 = +0.49$: The condition (14) will then give the limit of stability to be $k_1 = -0.372$ (i.e. $\sqrt{k_1} = 0.61i$) which again means that the relative decrease of the inward directed perturbative force must be $\left(\frac{0.862}{mP_0/m\omega^2 R}\right)\frac{y}{R}$ when y is the increase of the orbital radius. An outward directed force (i.e. P_0 negative) increasing by the same amount would also make a resonant extraction of the particles possible. Fig. 5 shows some increasing oscillations of particles very near to the stability limit. The wave-length of the free oscillations is close to twice the circumference of the orbit.

Fig. 6 displays the growth of the amplitudes for various initial conditions and Fig. 7 shows how the stability-limit increases when the azimuthal range of the force is decreased.

The condition (14) allows us to calculate regenerative extraction devices where quite suddenly an extraction force is applied or where the particle suddenly enters into a field where such a force is acting.[4]

Especially with smaller accelerators where the time for a revolution of the particles is short, it is not always possible to apply the extraction force suddenly, i.e. in a time which is short as compared to the duration of a revolution. In such cases, we might use an extraction force increasing relatively slowly with time and we will now see what influence this might have on the orbital stability of the free oscillations.

Such a case is shown in Fig. 8. After each revolution, the extraction force has increased by a certain amount thus creating new initial conditions for the closed orbit (y_e) as well as for the free oscillations (y_f). The calculation is somewhat lengthy but follows strictly the method described before. Fig. 9 shows the maximum amplitudes of the resulting free oscillations in two cases; with $k_1 = k_2 = +0.49$ and for $k_1 = -0.36$ and $k_2 = +0.49$, the near resonant case of Fig. 5 and 6.

Our calculations for the latter case show no sign of any resonant increase of the free oscillations. Obviously, the steadily increasing perturbative force has destroyed the conditions necessary for such resonances. A resonant extraction is, therefore, only possible when the perturbation force has reached a constant value.

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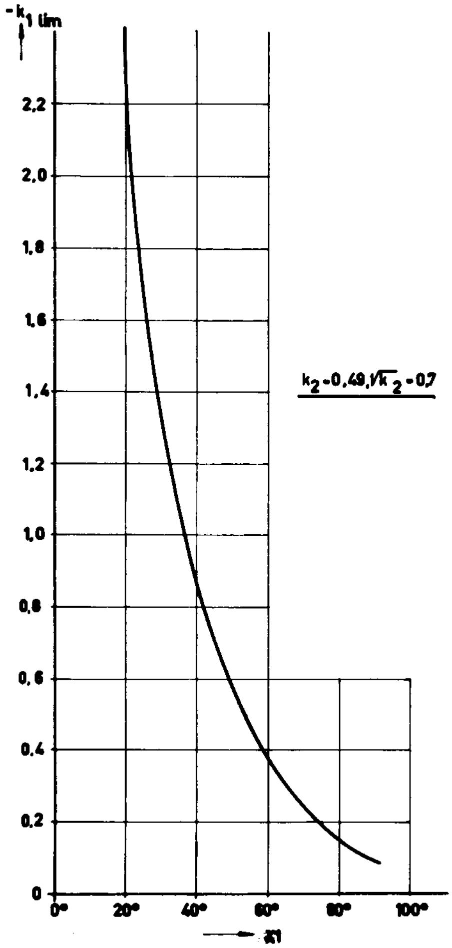

Fig. 1 Closed orbit (B) produced by a constant inwardly directed perturbation force P_1 acting over the azimuthal angle $2x_1$. A-A is the undisturbed orbit.

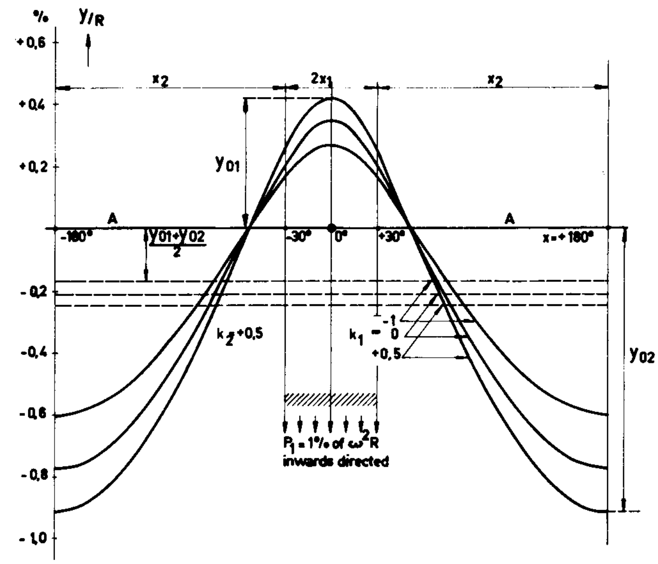

Fig. 2 Closed orbits produced by a radius-dependent azimuthally constant perturbation force P_1 with the frequency-coefficients k_1 = +0.5, 0 and −1. A-A = undisturbed orbit.

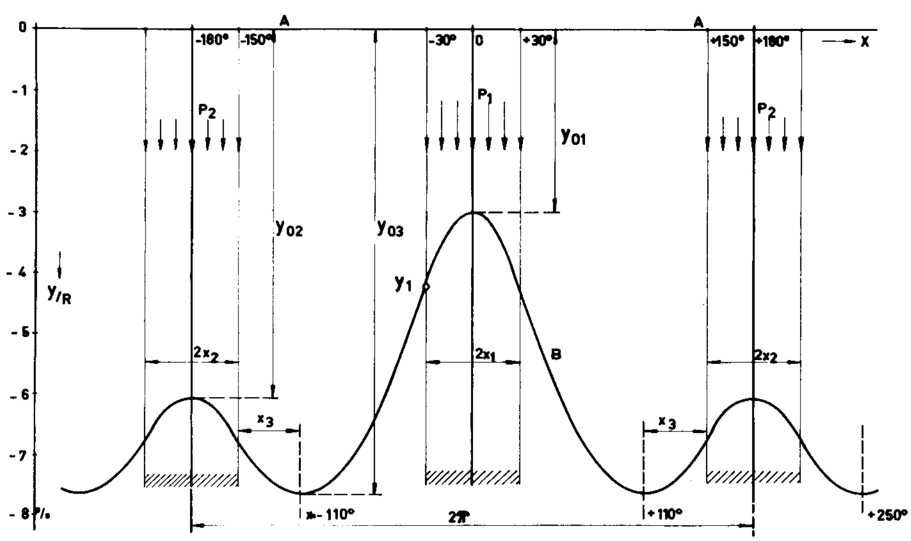

Fig. 3 Closed orbit produced by two azimuthally constant inwardly directed forces P_1 and P_2. A-A = undisturbed orbit.

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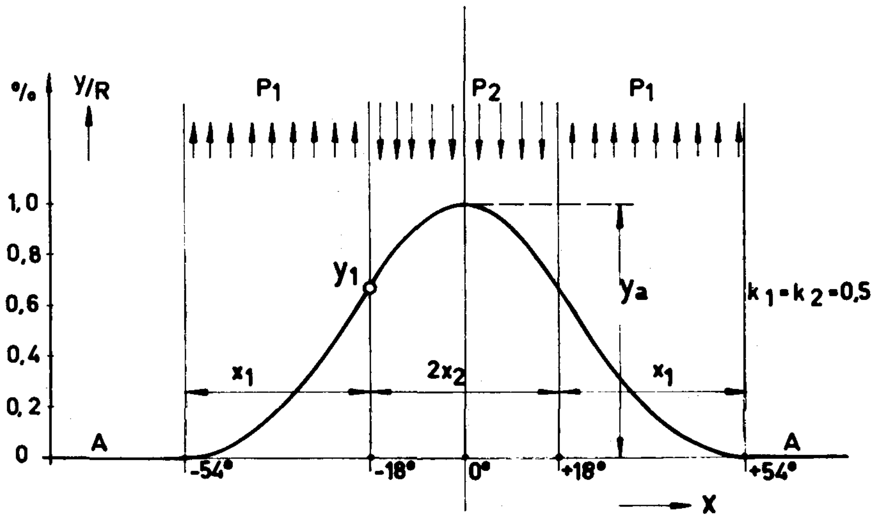
Fig. 4 Creating a "bump" in the closed orbit (B) by means of three azimuthally constant forces without disturbing the rest of the circular orbit.



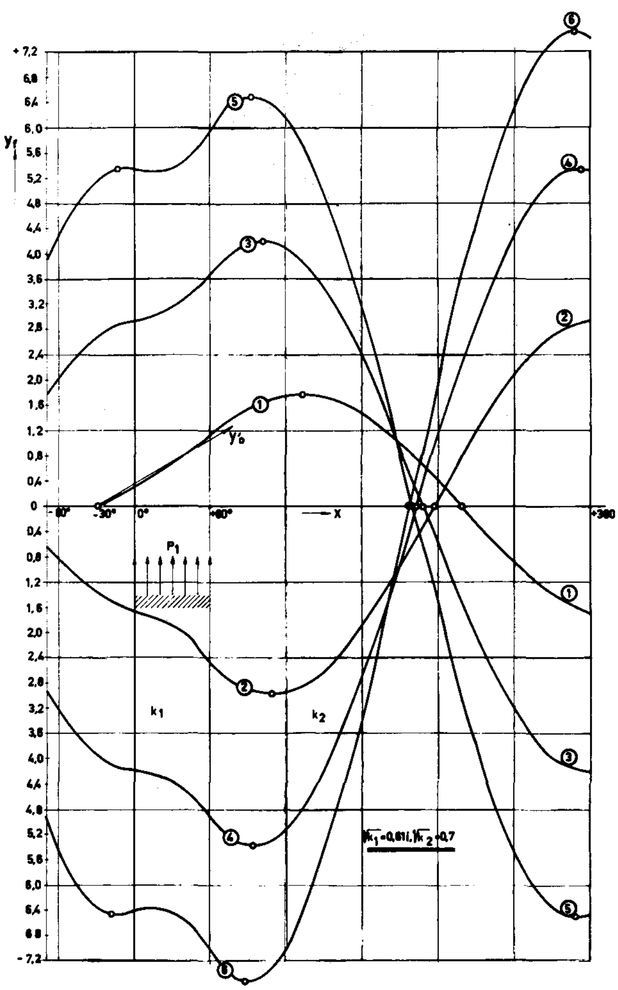


Fig. 5 Increasing free oscillations when the system is very close to the limit of stability. A-A = closed orbit.

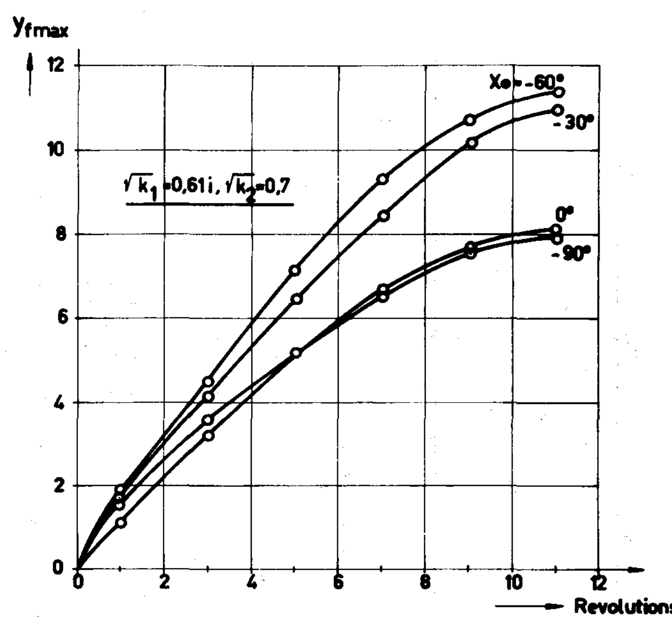


Fig. 6 Growth of the amplitudes of the free oscillations as shown in Fig. 5 for various initial conditions.

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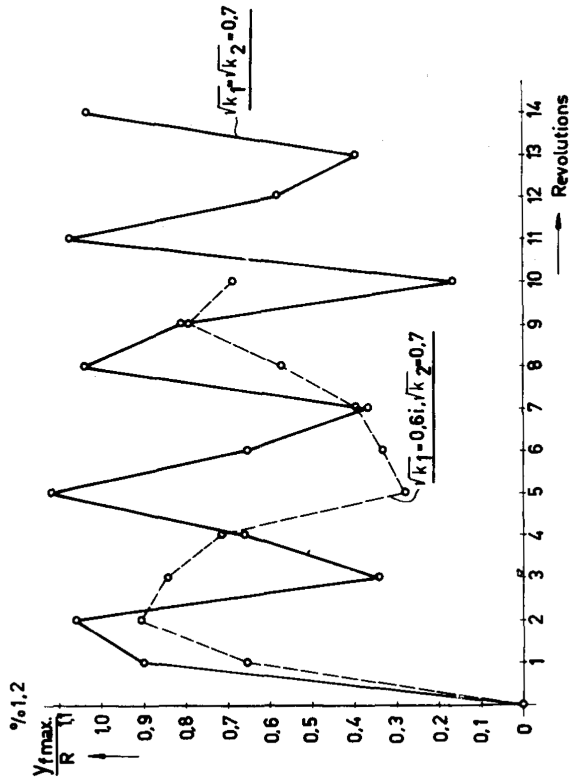

Fig. 9 Amplitudes for the free oscillations with a steadily increasing perturbation force.

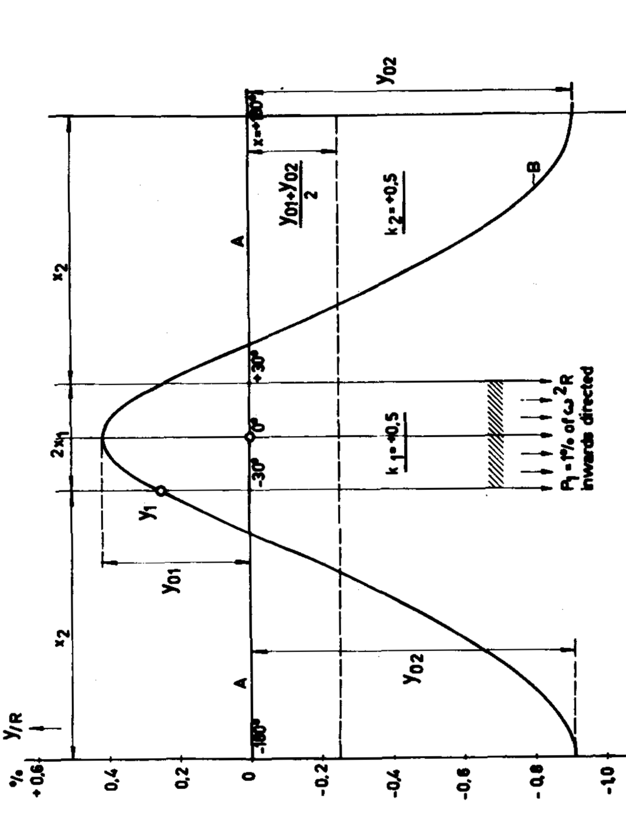

Fig. 7 Variations of the resonant limit for k_1 (with $k_2 = +0.49$) with the azimuthal range of the perturbation force.

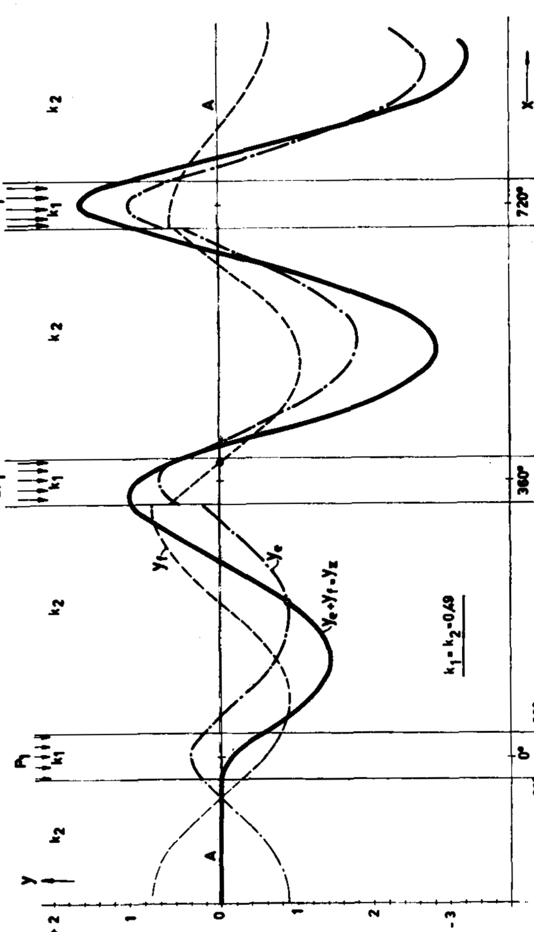

Fig. 8 Oscillations with a steadily increasing perturbation force.