THEORETICAL STUDY OF BEAM EXTRACTION BY THE METHOD OF CURVILINEAR COORDINATES

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This paper deals chiefly with this problem : Given the magnetic field of a cyclotron, what initial conditions of position and velocity are to be given to a particle in order that it should be ejected? A second question will be more briefly treated : In case the ejection occurs, at what azimuth does it take place?

I will restrict myself here to the study of median plane trajectories, thus leaving out of account the problem of vertical focusing. The method used rests basically on the choice of convenient curvilinear coordinates, the possibilities of which can be developed in two directions : either to find out the shape of stability boundaries by the means of topological considerations, or to get approximate solutions by an iteration process.

1. Curvilinear Coordinates and Equations of Plane Motion

Let the set of equilibrium trajectories C be plotted (Fig. 1) and extended sufficiently beyond the maximum energy radius. Each point of the plane is defined by its coordinates R, α, where R is the average radius of C and α the azimuth of the normal to C. Let $e = \frac{\delta n}{\delta R}$ be the ratio of the normal distance between two neighbouring trajectories to the difference of their mean radii; ρ the radius of curvature of C; and a the radius of curvature of the orthogonal curves to the trajectories C. One then has $\frac{1}{a} = -\frac{1}{\rho e}\frac{\delta e}{\delta \alpha}$, where e, ρ, a are functions of R, α, periodic in α with period $2\pi/N$.

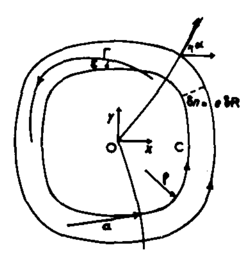

Fig. 1

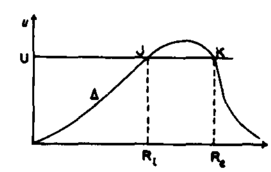

Fig. 2

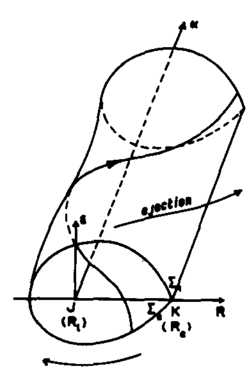

Fig. 3

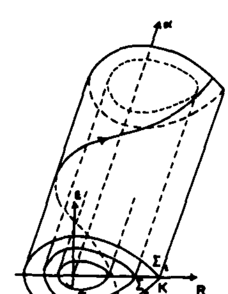

Fig. 4

The energy of a particle on an equilibrium trajectory is defined by its velocity v or, which is better, by its velocity related to the proper time $u = v/\sqrt{1 - v^2c^2}$ (= p/m_0). u is a function of R only, the behavior of which is given by the curve Δ in Fig. 2.

Let Γ be a given arbitrary non-accelerated trajectory, with given velocity U (V in usual time). The line U = C t cuts the curve Δ in two points of abscissas R_i and R_e, to which correspond two equilibrium trajectories C_i (internal) and C_e (external). As U is increasing from zero, the radius $R_i(U)$ of the equilibrium trajectory climbs the left slope of Δ.

The set of equilibrium trajectories and the law of velocity u(R) specify the magnetic field H, due to the relation $1/\rho = qH/m_0u$.

A plane trajectory Γ is defined by the functions R(S) and α(S), where S is the length along the trajectory. One defines (Fig. 1) the angle ε of Γ with the C it crosses. Let E_N and E_U be the components of the electric field normal and tangent to Γ, and $h = \eta H$ the extra field of any magnetic bump. The system of differential equations for the plane motion is then

$$\frac{dR}{dS} = \frac{\sin\epsilon}{e}, \quad \frac{d\alpha}{dS} = \frac{\sin\epsilon}{a} + \frac{\cos\epsilon}{\rho},$$

$$\frac{d\epsilon}{dS} = \frac{\sin\epsilon}{a} + \frac{\cos\epsilon}{\rho} - \frac{1}{\rho}\frac{u}{U} + \frac{q}{m_0}\frac{E_N}{UV} - \frac{1}{\rho}\frac{u}{U}\eta, \tag{1}$$

$$\frac{dU}{dS} = \frac{q}{m_0}\frac{E_U}{V}.$$

If $E_U = 0$, U is a constant and the system reduces to its first three equations.

2. Ejection Theorem for Free Oscillations

"Free" means here the absence of any electric field or magnetic bump, and accordingly the velocity U is constant. The system (1) becomes

$$\frac{dR}{dS} = \frac{\sin\epsilon}{e}, \quad \frac{d\alpha}{dS} = \frac{\sin\epsilon}{a} + \frac{\cos\epsilon}{\rho},$$

$$\frac{d\epsilon}{dS} = \frac{\sin\epsilon}{a} + \frac{\cos\epsilon}{\rho} - \frac{1}{\rho}\frac{u}{U}. \tag{2}$$

One can state :

The necessary and sufficient condition for a particle of fixed velocity U to be ejected is that its radius R reaches the value R_e with an angle ε positive,

and inversely :

The necessary and sufficient condition for a particle which is moving outwards to be repelled inwards (at least for a while) is that ε becomes zero before R has reached R_e. As long as the motion remains inside C_e, the successive minima of R are smaller than R_i and the successive maxima lie in the interval (R_i, R_e).

This theorem concerns only "centred" trajectories, meaning trajectories which have the cyclotron centre on their concave side.

3. "Cage" Surfaces

We will use in the following a representation in the R, ε, α space.

Consider for a fixed velocity U the trajectories C_i and C_e of radii R_i and R_e; they are straight lines in the R, ϵ, α space (see Fig. 3). C_i may be stable or unstable.

It can be shown that the trajectory C_e is always unstable. In the linear approximation for betatron oscillations about C_e there exist two one-parameter families of motion

$$\Delta R_1 = A\varphi_1(\alpha)\exp(-\nu\alpha) \quad , \quad \epsilon_1 = A\frac{e}{\rho}(\alpha)\left(-\nu\varphi_1 + \frac{d\varphi_1}{d\alpha}\right) \quad ,$$

$$\Delta R_2 = B\varphi_2(\alpha)\exp(+\nu\alpha) \quad , \quad \epsilon_2 = B\frac{e}{\rho}(\alpha)\left(\nu\varphi_2 + \frac{d\varphi_2}{d\alpha}\right) \quad , \qquad (3)$$

where $\Delta R = R - R_e$; $\nu > 0$; φ_1 and φ_2 periodic with period $2\pi/N$ and with definite signs; e/ρ is evaluated along C_e; and A and B are arbitrary constants. The first motion vanishes as $\alpha \to +\infty$, the second as $\alpha \to -\infty$.

From this we may infer that also for the exact system (2), there exists a one-parameter family of trajectories which tend towards C_e as $\alpha \to +\infty$, and another family which tends towards C_e as $\alpha \to -\infty$. They are asymptotic to the motions in the linear approximation (3). In the R, ϵ, α space these trajectories generate two surfaces Σ_1 and Σ_2 which have the straight line $R = R_e$ in common. We will consider only the left parts of these surfaces ($R \le R_e$). Both Σ_1 and Σ_2 are periodic in α with period $2\pi/N$. They are cylinder-like surfaces, and cut the R, α plane at abscissas smaller than R_i. To build these surfaces it is simpler, rather than using the two families of asymptotic trajectories, to consider ϵ as a function of R, α obeying the partial differential equation

$$\frac{\sin\epsilon}{e}\frac{\delta\epsilon}{\delta R} + \left(\frac{\sin\epsilon}{a} + \frac{\cos\epsilon}{\rho}\right)\frac{\delta\epsilon}{\delta\alpha} = \frac{\sin\epsilon}{a} + \frac{\cos\epsilon}{\rho} - \frac{1}{\rho}\frac{u}{U} \quad , \qquad (4)$$

for the boundary conditions at Σ_1 : $\epsilon = 0$ and $\left(\frac{d\epsilon}{dR}\right)_1 = \frac{e}{\rho}\left(-\nu + \frac{1}{\varphi_1}\frac{d\varphi_1}{d\alpha}\right)$ for $R = R_e$, and at Σ_2 : $\epsilon = 0$ and $\left(\frac{d\epsilon}{dR}\right)_2 = \frac{e}{\rho}\left(\nu + \frac{1}{\varphi_2}\frac{d\varphi_2}{d\alpha}\right)$ for $R = R_e$. This constitutes a special case of Cauchy's problem.

For the case that the set of real (meaning in the median plane, not in the R, ϵ, α space) equilibrium trajectories has a symmetry axis, the two surfaces Σ_1 and Σ_2 will join tangentially in the R, α plane with vertical tangent planes to form a single surface Σ. The junction line (periodic in α) is entirely to the left of R_i, since the real trajectories enter more inwardly than C_i.

When the set of real equilibrium trajectories has no symmetry axis (as for spiral ridge sectors), one may consider this set as derived from a symmetrical one by the variation of a parameter; and, if one admits that the solution of Eq. (4) depends continuously on this parameter, the two surfaces Σ_1 and Σ_2 will still join as in the case of symmetry.

As above, these results concern only trajectories of sufficiently large energies to be centred.

The tube-like surface Σ defines two regions in the R, ε, α space, and no trajectory except those which lie on Σ can cross Σ. Hence, the interior of this surface is the non-ejection zone, and any particle following a trajectory originating from a point outside Σ is ejected. We call Σ the "cage" surface, to distinguish it from stability boundaries, which might, at least a priori, exist inside or outside Σ without any connection ejection.

The behavior of the trajectories is particularly simple in the case of an azimuthally uniform field. The Eq. (2) becomes

$$\frac{dR}{dS} = \sin \epsilon \quad , \quad \frac{d\alpha}{dS} = \frac{\cos \epsilon}{R} \quad , \quad \frac{d\epsilon}{dS} = \frac{\cos \epsilon}{R} - \frac{1}{R}\frac{u}{U} \quad , \tag{5}$$

where R, α now are the ordinary polar coordinates. This system is integrable (Stormer's invariant), with the solution

$$R \cos \epsilon - \Phi(R) + K = 0 \quad , \quad \alpha = \int \cot \epsilon \frac{dR}{R} = \int \frac{\Phi - K}{\pm\sqrt{R^2 - (\Phi - K)^2}} \frac{dR}{R} \; . \tag{6}$$

Here K is a constant and $\Phi(R) = \int_{R_0}^{R} \frac{u(R)}{U} dR$ with R_0 arbitrary. This leads to the surface Σ of Fig. 4, the cross-section of which is given by Eq. (6) for $K = \Phi(R_e) - R_e$; any trajectory inside Σ lies on a tube T characterized by a value $K > K_i = \Phi(R_i) - R_i$ (the value for C_i) and $< K_e = \Phi(R_e) - R_e$.

4. Extraction

The purpose of any extracting device is of course to displace the R, ε, α conditions from the inside to the outside of the surface Σ. Let us consider what the above theoretical considerations imply on the means for doing this.

The effect of the RF acceleration is gradually to shrink the surface Σ as U rises (Fig. 2). Even if C_i remains stable, this shrinking causes the inner points near Σ, corresponding to real trajectories of large amplitudes, to traverse Σ and hence be ejected. Moreover, C_i may become definitly unstable. In this case, for a given energy, the inner trajectories are probably unrolling asymptotically towards Σ, and this obviously enhances their tendency to cross Σ during the acceleration.

A fast extraction will occur when by some deflecting device the particle is given a sufficiently large kick in R, ε, α for the point J itself suddenly to cross Σ (Fig. 4).

Let us now examine the case of repeated smaller kicks (electric deflector or magnetic bump) described by the last two terms of the third Eq. (1). The process can be easily followed for non-accelerated particles in an azimuthally uniform field. Each kick causes a change in the value of the constant K of Eq. (6), and the R, ε, α point thereby to pass from one tube T to another. If the perturbation per turn is small enough (as by regenerative extraction), K increases monotonicly from K_i, reaches and exceeds K_e, whereby the ejection is accomplished. If the perturbation is suitably small, K undergoes a series of oscillations, the extrema of which always occur at practically the same azimuth with the maximum or the minimum at the mean azimuth of the deflector. Then two possibilities can occur, presumably according to the form of the deflecting field : a) There is no trend in the oscillations of K, and the R, ϵ , α trajectory lies from time to time on a tube very near Σ, thus providing a sort of storage. The maximum radius ($\sim R_e$) is always reached at the same azimuth, and it is theoretically possible to give the particle a supplementary kick to extract it without using any septum. If in particular the perturbing field is quite small, as will be the case in the interior of the cyclotron, where the action of the deflector vanishes, there is an adiabatic effect by which the oscillations of K cancel. b) The oscillations of K do have a trend, and the maximum of R approaches smoothly its extraction value R_e, always at the same azimuth.

These results are obviously connected with the phenomena of resonant extraction[1)], spill beam[2)], and experiments on beam debunching in FM cyclotrons[3)]. The inclusion of RF acceleration, as well as field modulation, will complicate the results, but probably not essentially.

5. Example of Approximate Solution : Fast Extraction

Consider Eq. (1) where we set $\eta = 0$, $\frac{q}{m_0}\frac{E_N}{UV} = f(R, \alpha)$, and assume E_U negligible with U still a constant. An approximate method of solution, based on Picard's process of iteration, is the following : we neglect in a first step the influence of the field modulation in e, ρ, α, replace $E_N(R, \alpha)$ by an average function $\bar{E}_N(R)$, and hence $f(R, \alpha)$ by an average function $\zeta(R)$. Eq. (1) then reduces to Eq. (5) with the extra term $\zeta(R)$ in the last equation. This system is still integrable and yields

$$R \cos \epsilon - \Phi(R) + G(R) = Ct \quad , \quad \alpha = \int \cot \epsilon \, \frac{dR}{R} \, , \qquad (7)$$

where $G(R) = \int_{R_0}^{R} R\zeta(R)dR$. Away from the deflector, one suppresses G(R) and obtain again Eq. (6).

The ejection condition in Section 2 may be written in accordance with these equations, giving :

$$\frac{q}{m_0 UV} \int_{R_0}^{R_c} R\bar{E}_N(R)dR > R_0 \cos \epsilon_0 - R_e + \int_{R_0}^{R_e} \frac{u}{U} \, dR \quad ,$$

where R_0 relates to the entrance, and R_c to the exit of the deflector. Given the angular extent $\Delta\alpha$ of the deflector, R_c may be found from the second equation (7); for instance, if $\epsilon_0 = 0$, $R_0 = R_i(U)$, one obtains $\frac{R_i}{2} \left(\frac{q}{m_0 UV} \bar{E}_N R_i \Delta\alpha\right)^2 > \int_{R_i}^{R_e} \left(\frac{u}{U} - 1\right) dR$, a condition for the strength of the deflector.

Let us now denote the solutions of Eq. (7) by $R_1(S)$, $\epsilon_1(S)$, $\alpha_1(S)$. The second step of the approximation is obtained by introducing these on the right hand sides of Eq. (1). Then, by simple quadratures, new solutions $R_2(S)$, $\epsilon_2(S)$, $\alpha_2(S)$ may be found. Let us take $R_1 = R$ as the independent variable instead of S, and set $\rho = r(1 + \lambda)$, $e = 1 + \mu$. Then by Eq. (6) one has $\oint \lambda d\alpha = 0$ and $\oint \mu d\alpha = 0$ over a full equilibrium trajectory, and we get,

$$dR_2 = \frac{dR}{1 + \mu\,[R,\ \alpha_1(R)]}\ , \quad d\alpha_2 = \frac{d\alpha_1}{1 + \lambda(R,\ \alpha_1)} + \frac{dR}{a(R,\ \alpha_1)}\ ,$$

$$d\epsilon_2 = \frac{d\epsilon_1}{1 + \lambda(R,\ \alpha_1)} + \frac{dR}{a(R,\ \alpha_1)} + \left[f(R,\ \alpha_1) - \frac{\zeta(R)}{1 + \lambda(R,\ \alpha_1)}\right]\frac{dR}{\sin\epsilon_1}\ ,$$

which improves the approximation and yet can be treated analytically. In particular the first equation permits one to optimize the azimuthal position of the deflector by the variational equation $\delta_\alpha \int_{R_0}^{R_e} \frac{dR}{1 + \mu} = 0$, or $\int_{R_0}^{R_e} \frac{dR}{[1 + \mu(R,\ \alpha_1)]^2}\,\frac{\delta\mu}{\delta\alpha} = 0$.

This method can also be used for slow extraction; it gives some of the results stated in Section 4.

References

1) H.G. Blosser et al., Nucl. Instr. and Meth. 18-19, 488 (1962).
2) J.R. Richardson et al., Nucl. Instr. and Meth. 18-19, 41 (1962).
3) A. Cabrespine, CRAS 250, 2544 (1960), and J. Phys. 21, 332 (1960).
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