# DEPOLARIZATION PROCESS IN SECTOR-FOCUSING CYCLOTRONS(*)

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(Presented by L.C. Teng)

The depolarization of a polarized beam of charged particles during acceleration in a circular accelerator has been studied by Froissart and Stora[1] at Saclay, Garren[2] at Berkeley, Lobkowicz and Thorndike[3] at Rochester, Courant[4] at Brookhaven, Cohen[5], and Khoe and Teng[6] at Argonne. Ref. 5 is a numerical study using an electronic computing machine; the other references treat the problem by various analytical approaches under different approximations. We shall present here a simplified analytical treatment following the approach of Ref. 6.

Since we are dealing with the average value or expectation value of the spin of a large ensemble of particles and since expectation values obey classical equations, we can employ the classical spin equation for this analysis. Relativistic covariant spin equations were given by Frenkel[7] in 1926 with the relativistic spin expressed as an antisymmetric 4-tensor, and by Bargmann, Michel, and Telegdi[8] in 1959 with the spin represented by an axial 4-vector. These equations are, of course, entirely equivalent (see, e.g., Ref. 6 for details). For an electromagnetic field which is approximately uniform in space so that the interactions between the field gradients and the magnetic moment of the particle are negligible, the spatial components of these equations give in the rest frame of the particle the usual Larmor equation

$$\frac{d\vec{s}}{d\tau} = \frac{ge}{2m}\vec{s}\times\vec{B}^{(R)} = \frac{e}{m}(1+G)\,\vec{s}\times\vec{B}^{(R)}\,, \tag{1}$$

and in the laboratory frame the equation

$$\frac{d\vec{s}}{dt} = \frac{e}{m}\left\{\frac{1}{\gamma}\,\vec{s}\times\vec{B}^{(L)} + \frac{1}{\gamma+1}\,\vec{s}\times\left(\vec{E}^{(L)}\times\vec{v}\right)\right.$$

$$\left. + G\left[\vec{s}\times\vec{B}^{(L)} - \frac{\gamma-1}{\gamma v^2}\,\vec{s}\times\left(\vec{v}\,\vec{v}\cdot\vec{B}^{(L)}\right) + \vec{s}\times\left(\vec{E}^{(L)}\times\vec{v}\right)\right]\right\} \tag{2}$$

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where $\hbar = c = 1$,

$e$ = charge of particle,

$m$ = rest mass of particle,

$g/2$ = gyromagnetic ratio of particle

= ratio of precession frequency to revolution frequency. This is not the usual definition of the gyromagnetic ratio, but it is convenient here to define it this way.

$G = \frac{g}{2} - 1$ = anomalous gyromagnetic ratio,

$\vec{v}$ = laboratory velocity of particle,

$\gamma = \frac{1}{\sqrt{1-v^2}}$,

$t$ = laboratory time,

$\tau$ = proper time,

superscripts (R) and (L) denote laboratory-frame and rest-frame quantities respectively

and $\vec{s}$ is the spin of the particle defined as the spatial components in the rest-frame of either the spin antisymmetric 4-tensor of Frenkel or the spin axial 4-vector of Bargmann, Michel, and Telegdi. Note here that the spin of a particle is meaningful only in its rest-frame. Eq. 2 is actually a hybrid equation giving the laboratory-time development of the (rest-frame) spin, expressed in terms of laboratory components of the electromagnetic field.

When $\vec{E}^{(L)} = 0$, Eq. 2 becomes

$$\frac{d\vec{s}}{dt} = \frac{e}{m\gamma} \vec{s} \times \left[ (1 + \gamma G) \vec{B}_{\perp}^{(L)} + (1 + G) \vec{B}_{||}^{(L)} \right] \tag{3}$$

where $\vec{B}_{\perp}^{(L)}$ and $\vec{B}_{||}^{(L)}$ denote respectively the components of $\vec{B}^{(L)}$ transverse to and along the direction of motion. Eq. 3 is identical to that given by Froissart and Stora[1] and by Telegdi and Winston[9]. The appearance of $\gamma$ in the coefficient of $\vec{B}_{\perp}^{(L)}$ shows that the gyromagnetic ratio becomes anisotropic as the particle energy increases and the ratio for transverse field increases with increasing energy. This change of the gyromagnetic ratio with energy introduces the possibility that at certain energy the precession frequency may equal the frequency of some oscillatory component of the field experienced by the particle, thus producing a condition for resonant depolarization.

In a sector-field circular accelerator, the only component of the magnetic field which does not vanish when averaged over a particle trajectory is the axial component. For a particle moving in this field with spin pointing approximately in the axial direction, the predominant secular motion of its spin is a precession about this mean axial field. Clearly, this precession does not affect the axial polarization of the

particle. Furthermore, since this field component is transverse to the motion, the frequency of this precession is, according to Eq. 3, dependent on the energy of the particle. All other field components including those which influence the polarization are oscillatory with zero mean value; hence, they do not contribute large secular depolarization effects unless there is a resonance between the frequency of the precession about the mean axial field and the frequency of one of the oscillatory depolarizing field components.

In the laboratory-frame, we shall employ an orthogonal curvilinear co-ordinate system (the orbit co-ordinate system) where the y co-ordinate runs along the equilibrium orbit, the z co-ordinate points in the axial direction, and the x co-ordinate lies in the median plane and perpendicular to the equilibrium orbit. The depolarizing field components at the position of the particle are, then, the x and y components, which can be written in expanded forms about the equilibrium orbit

$$\left[\begin{aligned} B_x^{(L)} &= \frac{\partial B_x^{(L)}}{\partial z}\, z + \frac{\partial^2 B_x^{(L)}}{\partial x \partial z}\, xz + \ldots \\ &= \frac{\partial B_z^{(L)}}{\partial x}\, z + \frac{\partial^2 B_z^{(L)}}{\partial x^2}\, xz + \ldots \\ B_y^{(L)} &= \frac{\partial B_y^{(L)}}{\partial z}\, z + \frac{\partial^2 B_y^{(L)}}{\partial x \partial z}\, xz + \ldots \\ &= \frac{\partial B_z^{(L)}}{\partial y}\, z + \frac{\partial^2 B_z^{(L)}}{\partial x \partial y}\, xz + \ldots , \end{aligned}\right. \tag{4}$$

where x and z are displacements of the particle from the equilibrium orbit, and where use is made of the Maxwell equation curl $\vec{B}^{(L)} = 0$ in getting the second expression for each component. The differential coefficients are evaluated on the equilibrium orbit and have the periodicity of the sector. From the Floquet theorem we know that x and z possess, in addition to the sector periodicity, also their respective betatron oscillation periodicities; namely

$$\left[\begin{aligned} x &= \sum_{n=0}^{\infty} x_n \cos\left[\left(\nu_x + nN\right)\Theta + \alpha_n'\right] \\ & \qquad\qquad\qquad\qquad (n = \text{integer}) \\ z &= \sum_{n=0}^{\infty} z_n \cos\left[\left(\nu_z + nN\right)\Theta + \beta_n'\right] , \end{aligned}\right. \tag{5}$$

where N is the number of sectors, $\Theta$ is an angle variable which replaces y and is

defined by $\frac{dy}{d\Theta} = \frac{1}{2\pi}$ (length of equilibrium) = R = equivalent radius and $\alpha_n'$ and $\beta_n'$ are constant phase angles. Expanding the coefficients in Eq. 4 in Fourier series and substituting Eq. 5, we obtain

$$\left[\begin{array}{l} B_x^{(L)} = \sum_{k,\ell,m=0}^{\infty} A_{k\ell m} \cos\left[(kN \pm \ell\nu_z \pm m\nu_x)\,\Theta + \alpha_{k\ell m}\right] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (k,m = \text{integers},\ \ell = \text{odd integers}) \\ B_y^{(L)} = \sum_{k,\ell,m=0}^{\infty} B_{k\ell m} \cos\left[(kN \pm \ell\nu_z \pm m\nu_x)\,\Theta + \beta_{k\ell m}\right] \end{array}\right. \tag{6}$$

where $\alpha_{k\ell m}$ and $\beta_{k\ell m}$ are new constant phase angles. The lowest order terms in Eq. 4 give terms with $\ell = 1$ and $m = 0$ in Eq. 6. The frequencies of these field components relative to the orbit co-ordinate system are, therefore,

$$\omega^{(L)} = (kN \pm \ell\nu_z \pm m\nu_x)\,\Omega^{(L)}\ , \tag{7}$$

where $\Omega^{(L)}$ is the revolution frequency of the particle given by

$$\Omega^{(L)} = \left\langle \frac{d\Theta}{dt} \right\rangle = -\frac{e}{m\gamma}\left|\left\langle \vec{B}^{(L)} \right\rangle\right|, \tag{8}$$

with $\left\langle \vec{B}^{(L)} \right\rangle$ denoting the mean axial field. An oscillatory depolarizing field can be decomposed into two oppositely rotating components. Near a resonance the components which rotate in the same direction as the precession about $\left\langle \vec{B}^{(L)} \right\rangle$ may cause large secular depolarization; the other component may be ignored. This is analogous to inducing transitions between spin states by the process of nuclear magnetic resonance.

The frequency of precession about $\left\langle \vec{B}^{(L)} \right\rangle$, as given by Eq. 3, is $(1 + \gamma G)\Omega^{(L)}$. This should be reduced by the revolution frequency of the particle to give

$$\omega_p^{(L)} = (1 + \gamma G)\Omega^{(L)} - \Omega^{(L)} = \gamma G \Omega^{(L)} \tag{9}$$

for the precession frequency relative to the orbit co-ordinate system employed here. This is clear from the consideration that if the anomalous gyromagnetic ratio G is zero the precession should just follow the revolution of the particle and the spin would appear to stand still in the orbit co-ordinate system. The resonance condition is, therefore, $\omega_p^{(L)} = \omega^{(L)}$ or

$$\gamma G = kN \pm \ell\nu_z \pm m\nu_x \qquad (k,m = \text{all integers},\ \ell = \text{odd integers}). \tag{10}$$

The order of a resonance is defined as $\ell + m$, and is equal to the degree of the lowest contributing terms in the expansions Eq. 4 of the depolarizing field. Thus, we can expect higher order resonances to be relatively weaker. Furthermore, all the resonances given by Eq. 10 may be called intrinsic because they are excited by fields having intrinsic sector periodicity N. When field errors are present, resonances with even $\ell$ may be excited by mid-plane errors and resonances similar to Eq. 10 but with the kN term replaced by all other positive integers may be excited by sector errors. These imperfection resonances being excited only by field errors are expected to be weaker than the intrinsic resonances.

The degree of depolarization is most easily calculated in the rest frame. The mean axial field $\left\langle \vec{B}^{(L)} \right\rangle$ being transverse to the motion becomes $\left\langle \vec{B}^{(R)} \right\rangle = \gamma \left\langle \vec{B}^{(L)} \right\rangle$ when transformed to the rest frame by the Lorentz transformation. The frequency of the precession about $\left\langle \vec{B}^{(R)} \right\rangle$ in the rest frame is, then, by Eq. 1

$$\omega_p^{(R)} = -(1+G)\frac{e}{m}\left|\left\langle \vec{B}^{(R)} \right\rangle\right| = -\gamma^2(1+G)\frac{e}{m\gamma}\left|\left\langle \vec{B}^{(L)} \right\rangle\right|$$

$$= \gamma^2(1+G)\Omega^{(L)} = \gamma\left(\omega_p^{(L)} + \gamma\Omega^{(L)}\right) , \qquad (11)$$

where for the last expression we have substituted Eq. 9. Obviously, the same transformation applies also to the frequency of the rotating component of the oscillatory depolarizing field $B_x$ and $B_y$, and together with Eq. 7 gives

$$\omega^{(R)} = \gamma\left(\omega^{(L)} + \gamma\Omega^{(L)}\right) = \gamma\left(kN \pm \ell\nu_z \pm m\nu_x + \gamma\right)\Omega^{(L)} \qquad (12)$$

for this frequency in the rest frame.

In the rest frame we shall employ a co-ordinate system which follows the depolarizing field component rotating with frequency $\omega^{(R)}$. With respect to this co-ordinate system the spin Eq. 1 becomes

$$\frac{d\vec{s}}{d\tau} + \vec{\omega}^{(R)} \times \vec{s} = \frac{e}{m}(1+G)\,\vec{s} \times \left(\left\langle \vec{B}^{(R)} \right\rangle + \vec{b}\right) , \qquad (13)$$

where $\vec{\omega}^{(R)}$ is a vector in the axial direction with magnitude given by Eq. 12 and $\vec{b}$ denotes the depolarizing field perpendicular to the axial direction which, now, appears as a constant vector in the rotating co-ordinate system. Eq. 13 can be written as

$$\frac{d\vec{s}}{d\tau} = \frac{e}{m}(1+G)\,\vec{s} \times \vec{B}_e , \qquad (14)$$

where the effective field $\vec{B}_e$ is defined by

$$\vec{B}_e = \left( \left\langle \vec{B}^{(R)} \right\rangle + \frac{m}{e(1+G)} \vec{\omega}^{(R)} \right) + \vec{b} \quad . \tag{15}$$

Eq. 14 shows that in the rotating co-ordinate system $\vec{s}$ precesses about $\vec{B}_e$ with the frequency

$$\omega_e = -(1+G) \frac{eB_e}{m} = \omega_p^{(R)} \left[ (1-q)^2 + r^2 \right]^{\frac{1}{2}} , \tag{16}$$

where $q = \frac{\omega^{(R)}}{\omega_p^{(R)}}$ and $r = \frac{b}{\left| \left\langle \vec{B}^{(R)} \right\rangle \right|}$ .

Let us assume that at $\tau = 0$, $\vec{s}$ is parallel to $\left\langle \vec{B}^{(R)} \right\rangle$ (axial direction). In a proper-time interval $\tau$ the spin will describe an angle $\omega_e \tau$ about $\vec{B}_e$. Denoting the angles between $\vec{s}$ and $\left\langle \vec{B}^{(R)} \right\rangle$ by $\Psi$ and the angle between $\vec{B}_e$ and $\left\langle \vec{B}^{(R)} \right\rangle$ by $\alpha$, we have the geometrical relationship (see Fig. 1).

$$\cos \Psi = \cos^2\alpha + \sin^2\alpha \cos(\omega_e \tau) \tag{17}$$

with $\alpha$ given by $\tan \alpha = \frac{r}{1-q}$ . If $P_1$ and $P_2$ are the probabilities of finding $\vec{s}$ respectively parallel and antiparallel to $\left\langle \vec{B}^{(R)} \right\rangle$, we have

$$P_1 - P_2 = \cos \Psi \quad \text{and} \quad P_1 + P_2 = 1. \tag{18}$$

Together with Eq. 16 and Eq. 17 this gives

$$P_2 = \frac{1 - \cos \Psi}{2} = \sin^2\alpha \sin^2 \tfrac{1}{2} (\omega_e \tau)$$

$$= \frac{r^2}{(1-q)^2 + r^2} \sin^2 \tfrac{1}{2} \left[ (1-q)^2 + r^2 \right]^{\frac{1}{2}} \omega_p^{(R)} \tau \quad . \tag{19}$$

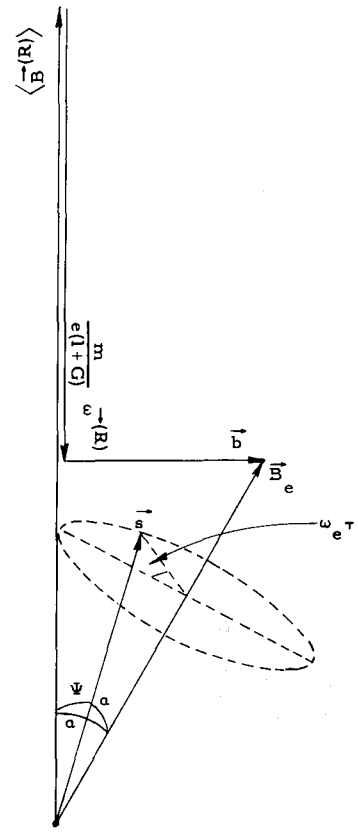


Fig. 1 Geometrical relationship of Eq. 17.

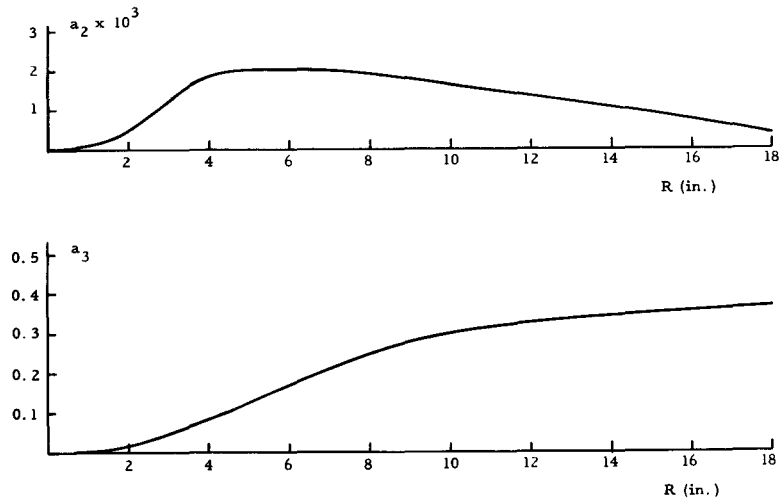


Fig. 2 Amplitudes of second and third harmonics of the axial field relative to its mean value.

The transition rate in the rest-frame from spin parallel to antiparallel is $\frac{dP_2}{d\tau}$ and the maximum depolarization is

$$P_2^{max} = \frac{r^2}{(1-q)^2 + r^2} \quad . \tag{20}$$

At resonance $\omega^{(R)} = \omega_p^{(R)}$, $q = 1$, and $P_2^{max} = 1$; i.e., we get complete depolarization.

In the discussion above, it was assumed that the effect of the oppositely rotating component of the oscillatory depolarizing field is negligible. From Eq. 20 we see that this assumption is indeed justified since near a resonance for this oppositely rotating component $1 - q \approx 2 >> r$ and, therefore, the increase of $P_2$ due to this component is entirely negligible.

For a numerical example, we give the data from the model measurement for the converted 54 in. 3-sector isochronous cyclotron at Washington University in St. Louis. The amplitudes of the second ($a_2$) and third ($a_3$) harmonics of the axial field relative to its mean value are plotted as functions of the radius R in Fig. 2. From the calculated values of $\nu_x$ and $\nu_z$ we find the following resonances to be relevant:

(1) The first-order imperfection resonance $\gamma G = 2-\nu_z$ will be encountered at $\gamma = 1.014$ (13 MeV) and R = 16 in. In the laboratory-frame, the second harmonics of the x and y components of the field at the position of the particle relative to the mean axial field are respectively

$$z \frac{\partial}{\partial x} [a_2 \cos (2\theta + \text{phase})] = z \frac{\partial a_2}{\partial R} \cos (2\theta + \text{phase}) ,$$

and

$$z \frac{\partial}{\partial y} \left[a_2 \cos (2\Theta + \text{phase})\right] = \frac{z}{R} \frac{\partial}{\partial \Theta} \left[a_2 \cos (2\Theta + \text{phase})\right]$$

$$= - z \frac{2a_2}{R} \sin (2\Theta + \text{phase}) \quad ,$$

where we used the relations $\frac{\partial}{\partial x} \approx \frac{\partial}{\partial R}$ and $\frac{\partial}{\partial y} = \frac{\partial}{R\partial\Theta}$. When transformed to the rest frame the amplitudes of these components become respectively $\frac{\partial a_2}{\partial R} a_z$ and $\frac{2a_2}{\gamma R} a_z$, where $a_z$ is the amplitude of the axial betatron oscillation. Therefore, the amplitude of the rotating depolarizing field component in the rest frame is given by

$$r = \frac{b}{\left|\left\langle \vec{B}(R) \right\rangle\right|} = \frac{1}{2}\left[\left(\frac{\partial a_2}{\partial R}\right)^2 + \left(\frac{2a_2}{\gamma R}\right)^2\right]^{\frac{1}{2}} a_z \ .$$

For $a_z = \frac{1}{2}$ in., this gives $r = 5 \times 10^{-5}$. Using Eq. 11 and noting that $\gamma\tau = t =$ laboratory time we can rewrite Eq. 19 as

$$P_2 = \frac{r^2}{(1-q)^2 + r^2} \sin^2 \left\{ \frac{\gamma}{2} \left[(1-q)^2 + r^2\right]^{\frac{1}{2}} (1+G)\Omega^{(L)} t \right\} \tag{21}$$

Substituting $q \approx 1$, $\gamma \approx 1$, $r = 5 \times 10^{-5}$, $(1 + G) = 2.8$ (for proton), $\Omega^{(L)} = 2\pi(20 \times 10^6) = 1.2 \times 10^8$ (revolution frequency = 20 Mc/s) we get

$$P_2 = \sin^2(8.8 \times 10^3)t \ .$$

Assuming an energy gain per turn of 140 keV, even if the resonance is 14 MeV wide and takes 100 revolutions or $t = \frac{100}{20 \times 10^6} = 5 \times 10^{-6}$ sec to cross, we will only get a depolarization of $P_2 = 0.002$, which is completely negligible.

(2) The second-order intrinsic resonance $\gamma G = 3 - \nu_z - \nu_x$ will be encountered at $\gamma = 1.006$ (5.6 MeV) and R = 10 in. Similar considerations give, in this case

$$r = \frac{b}{\left|\left\langle \vec{B}(R) \right\rangle\right|} = \frac{1}{2}\left[\left(\frac{\partial^2 a_3}{\partial R^2}\right)^2 + \left(\frac{3}{\gamma R} \frac{\partial a_3}{\partial R}\right)^2\right]^{\frac{1}{2}} a_x a_z$$

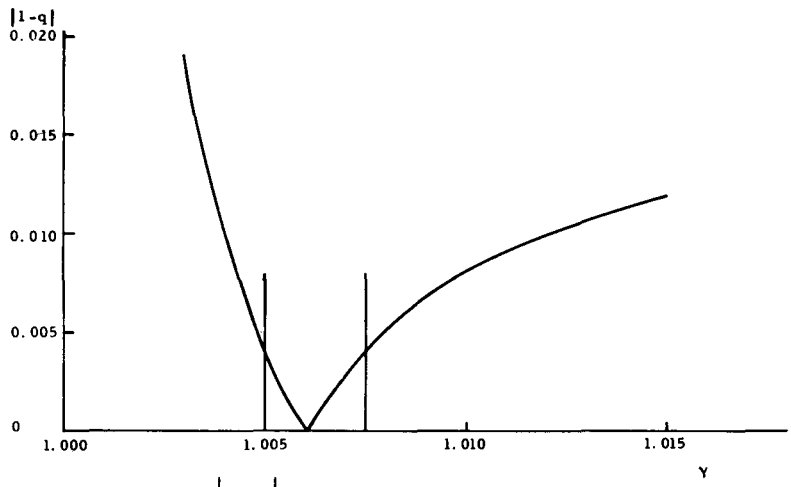


Fig. 3 The quantity $|1 - g|$ as a function of $\gamma$.

where $a_x$ and $a_z$ are the amplitudes of the radial and axial betatron oscillations respectively. For $a_x = a_z = \frac{1}{2}$ in. we get $r = 1.5 \times 10^{-3}$. In Fig. 3, the quantity $|1 - q|$ is plotted as a function of $\gamma$. To estimate the integrated degree of depolarization in crossing this resonance we assume that $q = 1$ for the range $1.005 \leqslant \gamma \leqslant 1.0075$, and that outside this range $|1 - q| >> r$ and the depolarization effect may be neglected. With an energy gain per turn of 140 keV, straightforward substitution in Eq. 21 gives $P_2 = 0.19$, which is certainly not negligible, but may still be tolerable.

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## DISCUSSION

KIM : We want to accelerate polarized deuterons in the Birmingham 40" cyclotron, so we need to calculate the depolarization phenomena. The definition of the anomalous part of the magnetic moment for the deuteron is very difficult. Application of the idea of the classical relativistic equation of spin motion shows that intrinsic resonance conditions for the deuteron can come out at energies of more than GeV. In the Birmingham cyclotron, two or three resonances can be produced by third harmonics of the magnetic field (main harmonics) for polarized protons in the low energy region. The total amount of depolarization can be 30%. What value did you take, for both the vertical and horizontal amplitude of the betatron motion?

TENG : I assumed $\frac{1}{2}$" oscillation amplitude. Actually there is one way of avoiding resonance, reducing it anyway, and that is just to have the vertical betatron oscillation amplitude as small as possible. In other words, inject as closely to the median plane as you can.