The Influence of Regenerative Extraction on the Energy Spread

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We were interested in the question of the quality of an external beam in a rather different situation from that just described Dr. Welton, namely, for a 150-Mev synchrocyclotron with an infinitely small dee voltage. Actually the dee voltage is of the order of 20 kv, giving an orbit spacing of less than 0.1 mm, while radial oscillations occur with amplitudes of 2 cm. In the following we will neglect the possibility of coherence between the oscillations of the individual particles. The particles are deflected by a regenerative deflection system. For this reason we have made a local field distortion (the regenerator) at $\theta = o$. The strength of the regenerator will be characterized by the function I(r), defined as

$$I(r) = \int \frac{\Delta B}{B} r \, d\theta \tag{1}$$

This definition is equivalent to a thin lens approximation of the necessarily extended field distortion.

In the unperturbed cyclotron field, i.e. from $\theta = +o$ to $\theta = 2\pi - o$ the particle will make the normal radial betatron oscillations with a frequency $\omega = \sqrt{1-n}$ around the equilibrium orbit $r = r_o$. The vector $\omega^{-1/2}(r - r_o)$, $\omega^{1/2} r'$, describing this oscillation in phase space, will rotate in one revolution of the particle in a clockwise direction through an angle of $2\pi\omega = 2\pi - \alpha$ with $\alpha = 2\pi(1 - \omega) \approx n\pi$. If we describe this in a coordinate system, rotating with unit frequency and coinciding with the original coordinates whenever $\theta = o$, we get a slow anti-clockwise rotation over an angle of α per revolution of the particle. We can describe this rotation by the equations

$$\frac{d(r - r_o)}{d\,m} = -\alpha\,\omega\,r'$$
$$\frac{d\,r'}{d\,m} = \alpha(r - r_o)/\omega \tag{2}$$

where m is the number of the revolution ($m = \theta/2\pi$), and where $r - r_o$ and r' represent the actual radial amplitude and its derivative with respect to θ only when $\theta = o$.

The effect of the regenerator is given by

$$\Delta r = 0 \qquad \Delta r' = -\,I(r)$$

or

$$\frac{d(r - r_o)}{d\,m} = 0$$
$$\frac{d\,r'}{d\,m} = -\,I(r) \tag{3}$$

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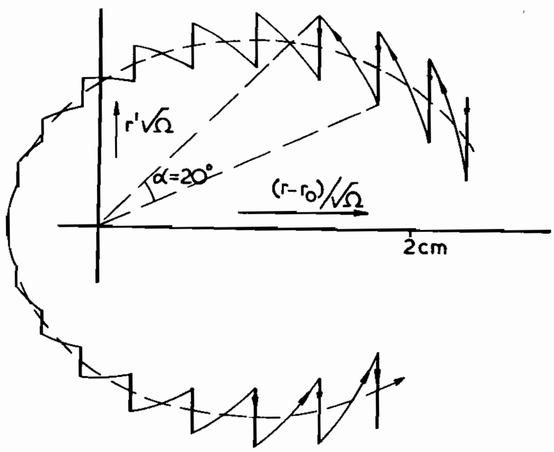

$I(r) = 0.2 + 0.15\,(r-r_0)$

Fig. 194. Successive revolutions in r, r' coordinates, represented in a coordinate system rotating with the cyclotron frequency. This system coincides with the original system each time $\theta = o$, i.e. when the particle is in the regenerator. The dotted line gives the smooth approximation.

The resulting orbit in phase space is shown in Figure 194. To solve the successive equations (2) and (3) for many revolutions we have made a smooth approximation of the orbit shown the figure by combining (2) and (3) into

$$\frac{d(r - r_o)}{d\,m} = -\,\alpha\omega r' \qquad\qquad (4)$$
$$\frac{d\,r'}{d\,m} = \alpha(r - r_o)/\omega - I(r)$$

To get an idea of the accuracy of these equations we have considered the case r_o = constant, $I = 0{,}2 + 0{,}15\,(r-r_o)$, and $n = 0{,}108$ (i.e. $\alpha = 20^o$). Then (2) and (3) can be solved exactly as I(r) is a linear function. The values of $(r-r_o)$ and r', measured in the center of the regenerator, are given by

$$r-r_o = 1.09 + A \cos 0.256\ m$$
$$r' = 0.698\ A \sin 0.256\ m$$

while the smooth approximation (4) gives

$$r-r_o = 1.11 + A \cos 0.258\ m$$
$$r' = 0.697\ A \sin 0.258\ m$$

The error corresponds to a very small change in n and I(r), without any significant effect on the expected results.

The phase lag α is relatively independent of the oscillation amplitude, in our case α remains practically constant for amplitudes up to 4 cm (r_o = 120 cm). The regenerator strength I(r) is a markedly nonlinear function, it does not seem likely, however, that this will effect the validity of the smooth approximation.

The equations (4) can formally be described as Hamilton equations following from a Hamiltonian

$$H(r,r';m) = -\frac{\alpha}{\omega}\left[\frac{1}{2}\,r'^2 + \frac{\omega}{\alpha}\int \alpha\,\omega\,(r-r_o) - I(r)dr\right] = \qquad (5)$$
$$= -\frac{\alpha}{\omega}\left[\frac{1}{2}\,r'^2 + V(r)\right]$$

This represents an oscillation in a potential well V(r), which is a slowly varying function of m, as r_o and, therefore, also α change because of the acceleration. In Figure 195 the lines $\alpha\omega\,(r-r_o)$ and I(r) are shown for a number of values of r_o. The difference between the lines is the integrand in (5). The assumed regenerator strength is represented by

$$I = 0.34\ (r-120\ \mathrm{cm}) + 0.07\ \mathrm{cm}^{-1}\ (r-120\ \mathrm{cm})^2$$

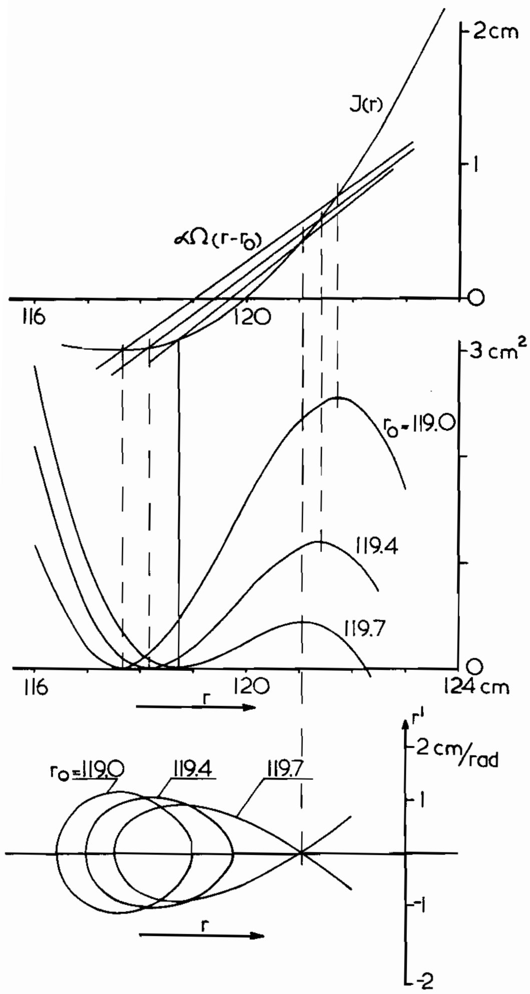

Fig. 195. Top: I(r) and $\alpha\,\omega\,(r-r_o)$.
Center: The potential well V(r) of Eq. 5.
Bottom: Lines H = constant for three values of r_o. The curves have the same area.

Figure 195 further shows the corresponding functions v(r). From the fact that H (r', r, m) changes adiabatically it follows that a group of particles, all lying on a curve H (r', r, m_1) = C_1 will some time later be found on a curve H (r', r, m_1) = C_2, where C_2 has to be determined so that the areas of both curves are equal. As r_o increases, the depth of the potential well decreases, so that the particles will eventually spill over. This is shown by the lower curves (Fig. 195). The curves H = C are shown here for particles which would have an amplitude of 1.23 cm at r_o = 120 cm if there were no regenerator.

The curves have thus an area of $\pi\omega$ (1.23 cm)^2. At r_o = 119.7 cm these particles have arrived on the separatrix and from then on the amplitude increases roughly exponentially until the particles leave the cyclotron. If we neglect the acceleration during this part of the deflection, we see that the momentum of particles with 1.23 cm radial oscillation amplitude corresponds to r_o = 119.7 cm while particles without oscillation amplitude will be ejected at r_o = 120 cm.

Figure 196 gives the relation between the oscillation amplitude and the final value of r_o calculated in this way. The energy spread, due to the radial oscillation, is thus quite small. For this it is essential that the curvature of I(r) near the final value of r_o be small as this will help to keep the potential well deep until r_o is quite close to its final value. The area in the r, r' space is, of course, conserved. The possibility of focusing the external beam both radially and axially will be determined, however, mainly by the distortions due to the nonlinearity of the fringing field and the field in the channel.

The foregoing analysis is a special case of nonlinear resonance, namely, a nonlinear integral resonance.*

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*R. Hagedorn, M. G. N. Hine and A. Schoch, CERN Symposium on High Energy Accelerators, p. 237 (1956).

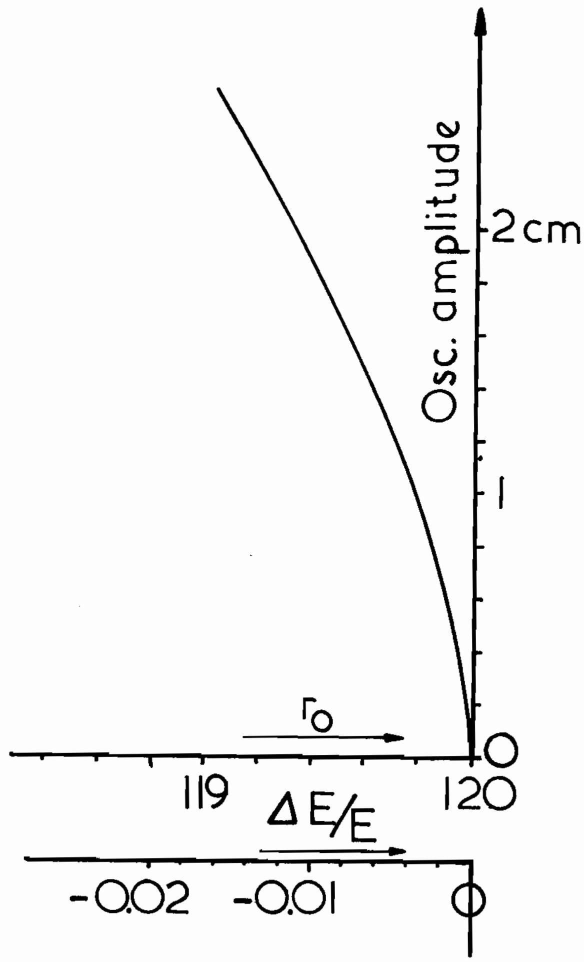

Fig. 196. The relation between the value of r_0 at the moment the deflection sets in and the original oscillation amplitude.

TENG: How does your calculated energy spread compare with the experimental data?

VERSTER: We have not yet measured the spread.

TENG: I made similar calculations for the synchrocyclotron. The calculated energy spread turns out to be much too large; the real energy spread they measure is much smaller.