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A Method of Computing for Nearly Periodic Orbits

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I must confess that I am a little overwhelmed at the undeserved credit that is being given to me for an old paper in which I was simply exploiting some of the methods that are in the astronomical literature for this physical problem of the orbits of particles in magnetic fields. You may say there is nothing new under the sun; certainly there is very little that the astronomers haven't discovered about numerical methods.

I want to ask whether some other methods that again are quite old might perhaps be of some use in the problems which have been talked about this morning and this afternoon. These are really somewhat technical matters of computing, but I think they can be explained fairly simply.

The basis of the integration of ordinary differential equations is to take points at a finite interval apart, and supposing you already have a number of points, you find at each of them the slope of the orbit and you use a number of backward values of the slope to predict a finite change to the next place in the orbit. The formula for doing this, or one of them, is often called Adams' formula. It actually was given by Adams in a certain paper, but you will find it attributed in various textbooks to Laplace, but Laplace attributes it to Lagrange(1). Don't always take unquestioned what people say about the sources of their formulas.

Well, this particular formula can be replaced by another which I believe is due to Gauss, although I am not sure. If, instead of taking places so that you have fairly small distance changes along an orbit, consider a case in which you have considerable distances after which the particle is nearly back in the same place as before. Such cases are found in various astronomical orbit calculations, particularly those of earth satellite orbits and to some extent at any rate in these cyclotron calculations. The successive places you get to after a definite number of revolutions may be fairly close together. If they are, it may be possible to extrapolate smoothly along many of these. If so, there is a formula for carrying out the integration ahead just like the Adams formula except that you replace the differential coefficient in the orbit by the whole displacement in one revolution which you calculate.

If you have this condition for successive steps of several revolutions ahead, you can use this process to extrapolate a large step of several revolutions. Then you calculate a single revolution and extrapolate again a large step of several revolutions.

We have found recently that this method, which does not work well for most astronomical cases, does work well for an earth satellite. You can get away in an earth satellite orbit calculation with calculating around the orbit at about one-minute intervals every tenth revolution and then going 10 revolutions ahead. The formula is just like Adams' formula except that it uses the small difference to give the large difference, instead of using the differential coefficient to give the large difference, and it is a type of formula that is well-known in the literature(2).

Whether this formula would have any application to the cyclotron case I don't know. It might have. What it will do for us is save, say, up to 90% of the actual computing. It does not make it possible to calculate an orbit you could not otherwise

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calculate. One can, however, go even further. Poincare showed that under certain circumstances you can get expansions in harmonic terms of motions in several variables, and if at each revolution you had a fairly large step it might still be possible, even though the equations are highly non-linear, to expand the result as a series in terms of suitable variables. If you do this the formulas that you finally get do not give you the long-time motion. If you use them for a small time ahead, there are formulas that can be used to step the coefficients of these full expansions. They are of the same general nature as the integration-ahead formula that I have just described, although the details have to be different and some of the equations that you get for integrating ahead are of an unstable character and have to be dealt with as what are sometimes known as stiff equations(3).

Using a finite Fourier analysis appears possible; using, say, only 12 or 24 terms might be sufficient for the astronomical problem of the combined motions of Jupiter and Saturn, where you are bothered by near commensurability, in their travel around the sun, which means that about every 5 revolutions of Jupiter you get back to a similar configuration. Whether such a method would be useful in the cyclotron problems again I don't know, but it seems that you can very often with these physical problems take advantage of the fact that the astronomers have been working at computing very much longer than the physicists.

CHAIRMAN JUDD: Thank you very much, Professor Thomas, for pointing out that the bag of tricks from the past has not been exhausted yet.

References

(1)F. Bashforth and J. C. Adams, Capillary Action, Cambridge University Press, Cambridge, 1883. See also, Laplace, Oevres, Gauthier Villars, Paris, Vol. X, 1894, p. 40.

(2)G. P. Taratynova, "The Motion of an Artificial Earth Satellite", The Russian Literature of Satellites, Part I, p 74, International Physical Index Inc., New York 1958, adapts the Runge-Kutta method for this purpose. Our formula, for ten steps ahead, runs

$$f_{50} = f_{40} + 10\,(f_{41} - f_{40}) + 4.5\,[(f_{41} - f_{40}) - (f_{31} - f_{30})]$$

$$+ 3.675\,[(f_{41} - f_{40}) - 2(f_{31} - f_{30}) + (f_{21} - f_{20})] + \text{etc.}$$

(next coefficients 3.2625, 3.0063875,...)
The coefficients depend on the number of steps.

(3)H. Poincare, Lecons de Mechanique Celeste, Gauthier Villars, Paris, 1905, Vol. I, p 171-6, 198, 268. See also, Laplace, Oevres, Gauthier Villars, Paris, Vol. 3, 1878, p. 256, and L. H. Thomas, Astronomical Journal, Vol. 63, 1959, p 433.