Calculating Orbit Properties of F-F Cyclotrons

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I am going to talk about some analytical results for calculating various orbit properties of fixed frequency cyclotrons. A great many people have worked on this, and there has been a good deal of discussion about how accurate they are and whether it pays to have more accurate formulas. I was not intending to talk about this, but I will in my talk try to point out where it is that the accurate formulas give better results and where it is that the formulas derived up to now by various people will serve just as well.

The first equation (Fib. 25) simply defines the magnetic field. For theoretical purposes we usually like to start out by assuming that the magnetic field is given in the median plane, and we give it in Fourier analysis form. The β_n are indications of spiraling. If you have no spiraling at all, no flaring, the β_n will be zero. The next equation simply gives the units used. The results are given in cyclotron units.

Starting with the third equation, the results are given, and I want to explain that the results are not given here in their most complicated and most accurate forms obtained; they are given in their more simple form, and there are more elaborate forms available. As it is, they are rather complicated.

$$H_z = H_0(r) + 2H_1(r)\cos[N\theta - \beta_1(r)] + 2H_2(r)\cos[2N\theta - \beta_2(r)] + \cdots$$

$$\frac{e}{m\omega} = \omega = c = 1$$

$$v = R$$

$$\nu_r^2 = E_0 + \frac{2g^2}{N^2 - 4E_0}$$

$$E_0 = \sqrt{1-R^2}\,(RH_0' + 2H_0) - 1 + \frac{2(1-R^2)}{N^2-1}\left\{R^2 H_1 H_1'' - R^2\beta_1'^2 H_1^2 + 4RH_1'H_1 + \frac{7}{4}H_1^2\right\}$$

$$g^2 = (1-R^2)\left\{\left(RH_1' + \frac{3}{2}H_1\right)^2 + R^2\beta_1'^2 H_1^2\right\}$$

Fig. 25. Definitions and results.

The first result simply is a connection between the velowity and the average radius of the equilibrium orbit, and to a fair approximation, they are equal. The next is an expression for the "tune." People have said quite a bit about the use of the smooth approximation, and I would like to point out that errors arising really have two origins. One is the use of the smooth approximation which breaks down here because the tune is near N/2, as Symon pointed out. The other is that terms have often been neglected in the equation which should not be neglected. This is not the most general result obtained, but it is the result that should apply to most cyclotrons being considered at present.

Figure 26 gives the expression for the axial tune. The E' term has two parts which are worth pointing out. The first term is the so-called Thomas term. The second term is the defocusing term due to the radial gradient. When these two terms very nearly cancel each other, the z/tune becomes difficult to calculate. This last equation is the

$$\nu_z^2 = E_0' + \frac{2f^2}{N^2 - 4E_0'}$$

$$E' = 2(1-R^2)(H_1^2 + H_2^2 + \cdots) - \sqrt{1-R^2}\, R H_0' - 2\,\frac{1-R^2}{N^2-1}\Big\{R^2 H_1'' H_1 - R^2 \beta_1'^2 H_1^2 + 2RH_1'H_1 + \frac{1}{4}H_1^2\Big\}$$

$$f^2 = (1-R^2)\Big[\big(RH_1' + \tfrac{1}{2}H_1\big)^2 + R^2 \beta_1'^2 H_1^2\Big]$$

$$H_0(r) = \frac{1}{\sqrt{1-r^2}}\Big\{1 + \Big(1 + \frac{1}{\nu_r^2}\Big)\frac{H_1^2}{N^2}\Big\} - \frac{2\sqrt{1-r^2}}{N^2}\big(rH_1'H_1 + 2H_1^2\big)$$

Fig. 26. Axial tune.

result for the magnetic field. It tells what the average magnetic field must be in order to give a constant going-around frequency. This is the average magnetic field as the function of r in terms of the harmonics and the tune.

Figure 27 gives some numerical results to indicate the accuracy of the formula. There are some errors in the calculations for the next two figures. However, the nature of the agreement with the theoretical results is still roughly as it appears here. This is the results on the Oak Ridge model cyclotron. On the left side, the radial tune is plotted, and on the right, the axial tune. The theoretical results given by the formulas just shown, or modifications of them, are given by the solid line, and the dotted line with the circles is the numerical results as computed. There are two things worth pointing out. One is the agreement at the lower end. The tune is nearly 1 there, but the difference from 1 is quite important, because of an adjacent resonance, and I think that in Blosser's report there is some actual data showing this point. This is one place where the old formulas usually break down. They do not give the difference from 1 very accurately, whereas the present formulas do give the difference from 1 rather accurately.

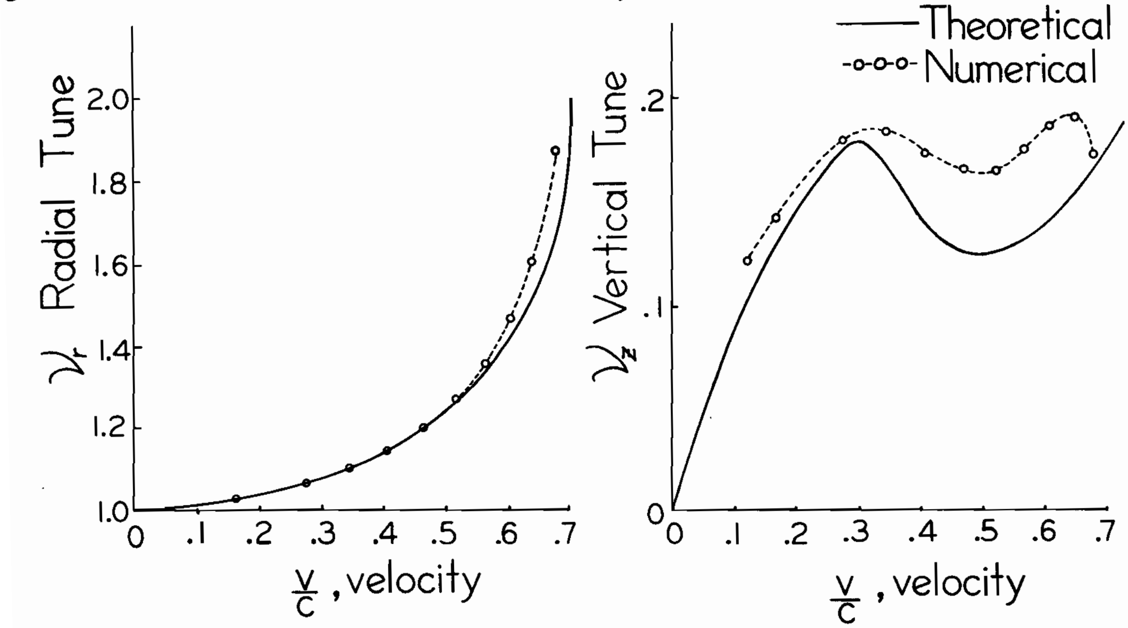

Fig. 27. Results for the Oak Ridge model cyclotron.

The other point of difference is the agreement at the upper end. Here you have a tune that is very close to one-half of N. (N is 4 here.) This is usually the case where the smooth approximation breaks down and the present formulas do fairly well here.

The agreement for the case of the z-tune is shown on the right side of the slide. In the central region, where the z-tune levels off, the Thomas term and the defocusing radial gradient term just about cancel each other, and the z-tune becomes difficult to compute as small terms in the equation become important. In some machines, the z-focusing is due to the spiraling. No great cancellation of terms occurs and the usually quoted formulas for the z-tune will do quite well. This may have been the case for Richardson's machine. The same sort of results for the California model are given in Figure 28.

I might say that these models are a rather severa test of theory because of the fact that they cover such a large range in tune, and they are rather peculiar fields. I would suspect that for the proton machines being considered, the agreement should be better.

Figure 29 gives the results of using the theory for calculating the average magnetic field. The first table compares the theoretical and numerical results for the average field required to give a constant going-around frequency. The second table compares the theoretical and numerical results for the amplitude of the oscillation of the equilibrium orbit about the average radius. No formula was given for this on the slides before, but formulas are available for this and all the other properties of equilibrium orbits and betatron oscillation orbits.

In Figure 30 are some results which perhaps are not too interesting. These are results which apply only in the very small region near the center, provided the field

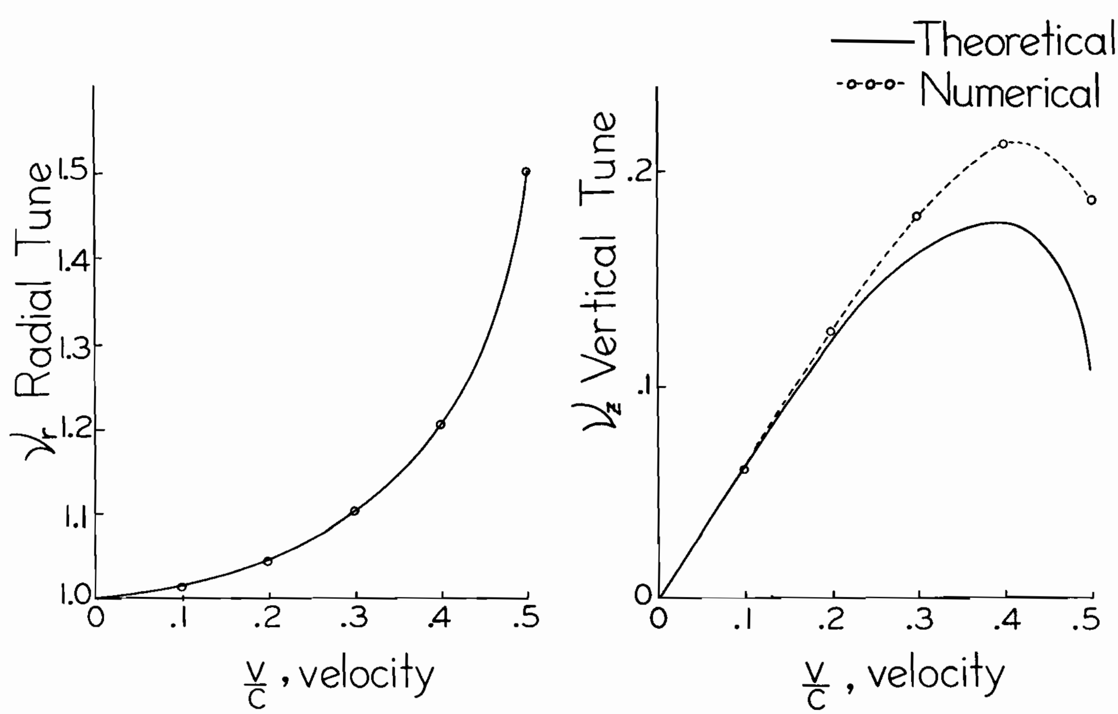

Fig. 28. Results for the California model.

r	$H_0(r)$	
	Numerical	Theory
0	1.0000	1.0000
.1	1.0032	1.0033
.2	1.0145	1.0149
.3	1.0348	1.0356
.4	1.0678	1.0694
.5	1.1136	1.1159
.6	1.1733	1.1786
.7	1.2586	1.2716

ν	$R_{max.} - R_{av.}$	
	Numerical	Theory
0	.0000	.0000
.1	.0007	.0009
.2	.0040	.0042
.3	.0088	.0090
.4	.0147	.0152
.5	.0225	.0240
.6	.0328	.0342
.7	.0470	.0489

Fig. 29. Numerical and theoretical results for Oak Ridge cyclotron.

$\underline{R \ll 1, \quad \nu \ll 1}$

$$H_0(r) = 1 + a_2 r^2 + \cdots$$

$$H_1(r) = b_1 r + \cdots$$

$$\beta_1(r) = \frac{1}{w} \ln r + \cdots$$

$$\nu_r = 1 + \frac{1}{2} R^2 \left\{ 4a_2 - 1 + \frac{b^2}{N^2-1} \left(\frac{37}{2} + \frac{25}{2} \frac{N^2-1}{N^2-4} \right) \right\}$$

$$\nu_z = R \left\{ 2 \left(b_1^2 \frac{N^2}{N^2-1} - a_2 + \frac{2b_1^2}{N^2} \frac{1}{w^2} \frac{N^2 - \frac{1}{2}}{N^2-1} \right) \right\}^{\frac{1}{2}}$$

$$a_2 = \frac{1}{2} - \frac{4}{N^2} b_1^2$$

Fig. 30. Results which apply near the center.

is accurately described by just an r^2 term in the average field and just an r term in the flutter field. In this region the results are particularly simple; that is why they are written down. The last result is just the conncetion between the average field and the flutter field for very small gradient. If you replace N^2 by $N^2 - 1$, you get a result that is supposed to be exact.

I want to make a few remarks about non-linear dynamics and then be done. In cyclotrons of the sort being considered today, there are roughly three non-linear resonances, which are due to non-linear terms in the equation when you expand around the equilibrium orbit, which may be of importance. There is a resonanct at $\nu_r = N/3$, which is particularly important if you have a three-sector cyclotron. Then there is a resonance at $\nu_r = N/4$, which arises if you have a four-sector cyclotron. Also there is the difference resonance, sometimes called the Walkinshaw resonance, $2\nu_z = \nu_r$. These are the three resonances which are likely to arise in cyclotron desing. I do not have any slides and so I just want to write down the results for the case of 1/3 resonance to show you what these formulas look like.

The stability limit amplitude, A_L, for the $\nu_r = N/3$ resonance is given by

$$A_L = \frac{4 \left| \nu_{ro} - 1 \right|}{B} ,$$

$$B = 1/2(1 - R^2)^{1/2} \left| R\, G_1'' + 4\, R\, G_1'' + 2\, G_1 \right| ,$$

$$G_1 = H_1\, e^{-i\beta 1}$$

where ν_{ro} is the radial tune for small amplitudes, and we assume N = 3. The tune ν_r varies with amplitude according to

$$\nu_r^2 = 1 + (\nu_{ro}{}^2 - 1)\ [1 - (\frac{A}{A_L})^2]^{1/2} .$$

When A exceeds A_L, ν_r becomes imaginary in the last equation and the imaginary part gives the exponential growth of the radial amplitude.

A program is being written at MURA which will evaluate the theoretical formulas. MURA also has a program called "Ill Tempered Five," proposed by F. T. Cole, which will do these calculations exactly, and which Symon will talk about later. There is no really final MURA report which contains the information in its final form, but I will write down what is available. There is a general report, MURA-397, which is on the theory of a general accelerator, and these results are based on the results in here. However, there are certain corrections which are not included in this report. There is a mimeographed note, of which I have copies, which contains the results on the slides.