BETATRON TUNE SHIFT DUE TO NONLINEAR R.-W. WAKE FIELD

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Abstract

A formula is presented for the coherent tune shift experienced by a single bunch traveling between two parallel resistive plates. It is shown that for the parameters of an LHC prototype collimator in the SPS, both the nonlinear wake-field components, calculated by Piwinski, and the correct time dependence, as derived, e.g., by Burov and Lebedev, must be taken into account. Similar considerations apply to the incoherent tune shift and to the stability diagram.

MOTIVATION

About 45 collimators represent the by far largest source of impedance in the LHC [1]. The LHC collimators consist of 1-m long thick graphite blocks, needed for survival in case of beam impact, and they are operated with half gaps as small as 1.5 mm. The collimator jaws are tapered over an additional 10 cm on either end, and they carry copper cooling tubes on their back. During the 2004 SPS run, beam measurements on a single collimator prototype were aimed at validating the impedance model [2]. The only impedance-related quantity which could clearly be resolved experimentally, by repetitive opening and closing of the graphite jaws, was the coherent tune shift.

THEORY VS. EXPERIMENT

Table 1 list parameters approximating the conditions of the SPS experiment (except that vertical and horizontal planes are exchanged) [3]. A vertical full gap of 2 mm corresponds to about 3 times the rms beam size σ_y, and to only 1.5 rms beam sizes in the orthogonal plane.

Table 1: Parameters.

bunch population	N_b	10^{11}
hor., vert. beta function	$\beta_{x,y}$	93, 25 m
dispersion function	D_y	0 cm
norm. transv. emittance	$\gamma\epsilon_{x,y}$	1.5 μm
rms hor., vert. beam size	$\sigma_{x,y}$	0.72, 0.37 mm
rms bunch length	σ_z	0.21 m
circumference	C	6912 m
vertical tune	Q_y	26.135
beam momentum	p	270 GeV/c
collimator half gap	b	1.5 mm
collimator thickness	d	30 mm
collimator resistivity	ρ	10 $\mu\Omega$m
collimator length	L	1 m

Comparing the experimental data with predictions from the classical resistive-wall (r.-w.) theory as described, e.g., by Chao [4], the measured tune shift is 2.5 times smaller than the expected one [3, 5] (see Fig. 3). Refined tune shift predictions are obtained from the Burov-Lebedev theory for flat chambers [6], which includes the effect of the finite chamber thickness and the so-called "inductive bypass," i.e., the correct limiting behavior at low frequency. However, a comparison of this theory with the measurement still reveals a factor of 2 discrepancy at small gaps. By contrast, at large openings, e.g., for full gaps of about 5 mm, the theory seems to match the experimental results (Fig. 3). This gives rise to the idea that for the smallest half gaps, which correspond to only 3 or 4 rms beam sizes, the nonlinear components of the wake field could be important.

The nonlinear wake potential, up to infinite order in the transverse positions of both drive (x_0, y_0) and probe particles (x, y), for the resistive-wall wake of a longitudinally Gaussian bunch passing between two parallel plates was derived by Piwinski [7] and re-written by Bane, Irwin and Raubenheimer [8]. At location $\tau = z/\sigma_z$ along the bunch (with $z > 0$ denoting positions in front of the bunch center), it is given by

$$V(x, y, x_0, y_0, \tau) = -\kappa f_R(\tau) \qquad (1)$$
$$\left[\frac{-\tilde{x}\sinh\tilde{x} + \bar{y}\sin\bar{y}}{\cosh\tilde{x} + \cos\bar{y}} + \frac{\tilde{x}\sinh\tilde{x} + \tilde{y}\sin\tilde{y}}{\cosh\tilde{x} - \cos\tilde{y}} \right] ,$$

with

$$\bar{y} = \frac{\pi}{2b}(y + y_0),\ \tilde{y} = \frac{\pi}{2b}(y - y_0),\ \tilde{x} = \frac{\pi}{2b}(x - x_0)\ , \quad (2)$$

$2b$ the full vertical gap between the plates, and

$$\kappa \equiv \frac{1}{2}\frac{N_b r_p}{\gamma\sigma_z}\frac{L}{b}\sqrt{\lambda\sigma_z} \qquad (3)$$

$$f_R(\tau) = \frac{\sqrt{2}}{\pi}\int_0^\infty \frac{d\tau'}{\sqrt{\tau'}}e^{-\frac{(\tau+\tau')^2}{2}} ,$$

where L is the length of the collimator, r_p the classical particle radius, N_b the bunch population, γ the relativistic Lorentz factor, $\lambda = \rho/(120\pi)$, and ρ the resistivity in $\mu\Omega$m. The average of f_R over a Gaussian bunch is $< f_R >_\tau = 0.816$. The Piwinski formula (1) applies, if the skin depth δ_s fulfills [7] $\delta_s \ll \min\left(d_{\mathrm{wall}}, d_{\mathrm{beam-wall}}, 1/(bk^2)\right)$, with d_{wall} the wall thickness, $d_{\mathrm{beam-wall}}$ the distance between the beam center and the wall, and $k = \omega/c$.

In (1), the time dependence and the dependence on the transverse coordinates factorize. We can therefore replace the time- (or frequency-) dependent part with the more precise expression of Burov-Lebedev, while keeping Piwinski's nonlinear transverse dependence, which represents a purely geometric effect.

In the Piwinski formalism, the vertical deflection $\Delta y'$ of a single particle is obtained as the negative derivative of

the potential with respect to the y coordinate of the probe particle, $\Delta y' = -\partial V/\partial y$.

We now assume that the effect of the collimator on a coherent oscillation is described by the centroid deflection

$$\Delta y'_c = -4\pi \Delta Q_y y_c/\beta_y + \mathcal{O}(y_c^2) , \qquad (4)$$

where $y_c \equiv\, <y>$ denotes the average vertical position of all particles, and $y'_c \equiv\, <y'>$ their average slope. We drop terms of higher order in the centroid position, and invert the above relation so as to obtain the coherent tune shift

$$\Delta Q_y = \frac{\beta_y}{4\pi}\left(-\frac{\partial(\Delta y'_c)}{\partial y_c}\right)_{y_c=0} . \qquad (5)$$

The expression in parenthesis is easily computed from the Piwinski theory, via

$$\frac{\partial(\Delta y'_c)}{\partial y_c} = \frac{\partial <\Delta y'>}{\partial y_c} = \left\langle \frac{\partial\left(\Delta y'(x,y,x_0,y_0)\right)}{\partial y_c}\right\rangle , \qquad (6)$$

assuming that the 4 transverse distributions in x, y, x_0 and y_0 are Gaussian, and that the two vertical ones are off-set from zero by a small coherent amplitude y_c. Two of the four integrations, over the horizontal and vertical distributions of probe and drive particles, respectively, can be performed analytically after a change of variables; a double integral is left for numerical evaluation. We obtain [3]

$$\Delta Q_{\mathrm{nl},y}^{\mathrm{flat}} \approx \frac{\beta}{4\pi}\frac{N_b r_p L}{2\pi\gamma} f_{\mathrm{nl}} \int_{-\infty}^{\infty} \frac{3}{2} Z_{\mathrm{BL},y}^{\mathrm{flat}}(\omega)\, e^{-\omega^2\sigma_z^2/c^2}\, d\omega . \qquad (7)$$

where $Z_{\mathrm{BL},y}^{\mathrm{flat}}$ refers to the Burov-Lebedev impedance [6]

$$Z_{\mathrm{BL},y}^{\mathrm{flat}} \approx -i\frac{\pi^2}{12}\frac{Z_0}{2\pi b^2}\frac{1}{1+\tau/2} , \qquad (8)$$

with $\tau \equiv \kappa b \tanh(\kappa d)$, $\kappa \equiv \sqrt{-4\pi i\sigma\mu\omega/c^2}$, and $|\kappa|b \gg 1$ is assumed. The complexity is hidden in the factor $f_{\mathrm{nl}}(b,\sigma_x,\sigma_y)$, which is defined as

$$f_{\mathrm{nl}} \equiv \left(\frac{2b^2}{\pi^2}\right)\frac{\mathrm{erf}\left(\frac{b}{\sigma_y}\right)}{\left(\mathrm{erf}\left(\frac{b}{\sqrt{2}\sigma_y}\right)\right)^2} \qquad (9)$$

$$\int_{-\infty}^{\infty}\int_{-2\pi}^{2\pi} G(X,Y)\frac{e^{-\frac{Y^2 b^2}{4\pi^2\sigma_y^2}-\frac{X^2 b^2}{4\pi^2\sigma_x^2}}}{4\pi\sigma_x\sigma_y}\, dY\, dX ,$$

where the error functions arise due to the distribution cut off at an amplitude equal to the half gap, and $G(X,Y)$ is

$$G(X,Y) = -\frac{1}{8}\left(\cos\left(\frac{Y}{2}\right)+\cosh\left(\frac{X}{2}\right)\right)^{-3} \qquad (10)$$

$$\left[\left\{2Y\sin\left(\frac{Y}{2}\right)-8\cos\left(\frac{Y}{2}\right)\right\}\cosh\left(\frac{X}{2}\right)^2\right.$$

$$-\{4\cos Y + 12 + Y\sin Y\}\cosh\left(\frac{X}{2}\right)$$

$$-4Y\sin\left(\frac{Y}{2}\right)-8\cos\left(\frac{Y}{2}\right)$$

$$\left.-X\left(\cos Y - 3\right)\sinh\left(\frac{X}{2}\right)+X\cos\left(\frac{Y}{2}\right)\sinh X\right] .$$

Figure 1 displays the function $G(X,Y)$ as a function of X for various fixed values of Y. The sign changes for large values of X. Also, G diverges for $Y \to 2b$ (not shown).
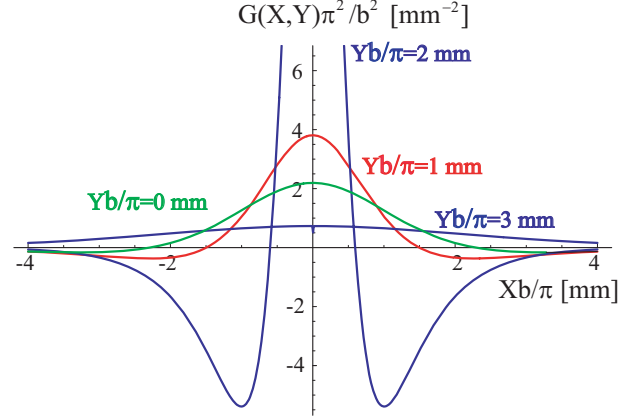

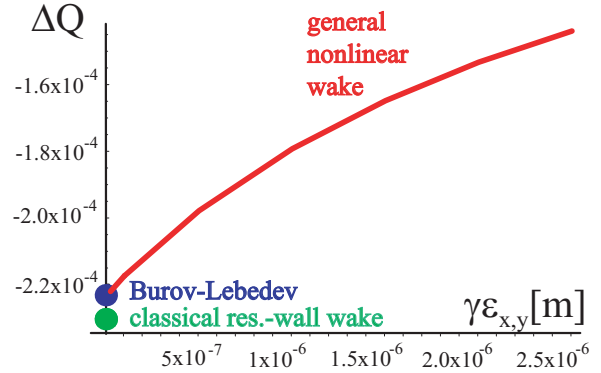Figure 1: $G(X,Y)$ vs. X for various values of Y.

Figure 2: Coherent vertical tune shift computed from the general nonlinear formula (7) as a function of the two transverse normalized emittances, for $\beta_x = 93$ m, $\beta_y = 25$ m, $\gamma \approx 288$, and a vertical half gap $b = 1.5$ mm. The linear estimate based on the flat-chamber Burov-Lebedev formula [6] and the one from the classical theory [4] are also indicated. As can be seen, the latter two theories are strictly applicable only in the limit of vanishing emittance, where the nonlinear contributions disappear.

The dependence of the coherent tune shift on the transverse emittance is illustrated in Fig. 2 for the parameters of Table 1. Figure 3 compares the coherent tune shift observed in the SPS experiment with the predictions from Chao's theory, Burov and Lebedev's theory, and the nonlinear wake-field formula (7). The latter three were calculated using the actual optical functions at the collimator, and, for each data point, the measured bunch intensity, bunch length, and, in the nonlinear case, also the transverse emittances, including error propagation [3]. Only the nonlinear formula (7) agrees well with the experimental data, while the other two theories deviate considerably.

Since a particle's deflection depends nonlinearly on its transverse and longitudinal coordinates, the collimator also
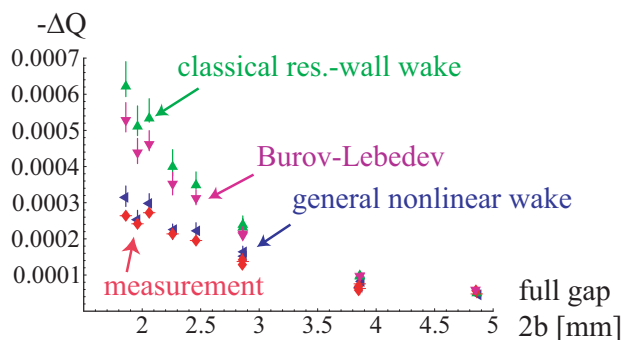
Figure 3: Measured data points from the SPS experiment [red rhombi] compared with predictions from classical theory [4] [light green upward triangles], Burov-Lebedev theory [6] [purple downward triangles] and general nonlinear theory (7) [blue sideward triangles]. The error bars on the predicted values represent both the statistical and the systematic errors.

induces an incoherent tune shift, the complete expression for which can be found in [3, 5]. Indeed, the actual experiment in the SPS exhibited some indirect evidence for a change in the tune spread: The natural oscillation amplitude was reduced when the collimator was closed [2, 9, 10], which could be explained by enhanced Landau damping.

OUTLOOK

The effects of nonlinear wake fields described here could prove important at other future accelerators. For example, the GSI FAIR project foresees the use of nuclotron-type pulsed s.c. dipoles, where the 3.5σ-beam envelope fills the entire beam-pipe aperture with an elliptical chamber of aspect ratio of 2:1 [11]. The nonlinear components of the resistive-wall wake field are likely to represent a significant effect.

An exact computation of the stability limit due to loss of Landau damping should include not only the coherent dipole wake, but also the incoherent and nonlinear wake components. As a first extension in this direction, we consider the effect of the incoherent "quadrupole" wake field. To this end, we write the equation of motion for a single particle as

$$\ddot{y}(t,\eta) + \omega^2(\eta)y(t,\eta) =$$
$$Y_1 \int_{-\infty}^{t} W(t-t')y(t',\eta)\lambda(t')dt'$$
$$+Y_2 \left(\int_{-\infty}^{t} W(t-t')\lambda(t')dt' \right) y(t,\eta) \quad (11)$$

where the first term on the right-hand side represents the conventional linear wake, and the second term the linear part of the incoherent wake; η is a parameter characterizing the frequency spread, λ the line density, and t the time; $Y_1 \approx \pi^2/12$ and $Y_2 \approx \pi^2/24$ are Yokoya factors [12] describing the relative differences with respect to a round-chamber wake field. Applying a Fourier-Laplace transform

to the above equation we obtain the dispersion relation including the incoherent wake field,

$$1 = Y_1 \tilde{W}_1 \int d\eta \frac{\rho(\eta)}{\omega^2(\eta) - Y_2 \tilde{W}_0 - z^2} , \quad (12)$$

where $z = (\omega_{coh} - i\alpha)$ [with α the growth rate], and

$$\begin{pmatrix} \tilde{W}_0 \\ \tilde{W}_1 \end{pmatrix} \equiv \frac{i}{T_0} \frac{N_b r_0 c^2}{\gamma C} \sum_{p=-\infty}^{\infty} \begin{pmatrix} Z_1^{\perp}(p\omega_0) \\ Z_1^{\perp}(p\omega_0 + \omega_{\beta}) \end{pmatrix} , \quad (13)$$

with T_0 the revolution period, $\omega_0 = 2\pi/T_0$, and ω_{β} the angular betatron frequency. According to (12), the incoherent linear part of the wake field only shifts the value of the coherent frequency, but it does not change the stability limit. However, a stabilizing effect of the collimator wake is likely to arise from the nonlinear wake-field components, not yet included in the dispersion relation.

SUMMARY

The nonlinear terms of the resistive-wall wake field become important if the aperture is comparable to the rms beam size. A generalized formula combining the Burov-Lebedev theory (dependence on ω) and the Piwinski theory (dependence on x, y, x_0 and y_0) is in nearly perfect agreement with the SPS measurement of coherent tune shifts. For small gaps, the incoherent tune spread from the nonlinear wake field may increase the beam stability via enhanced Landau damping.

Details on the various theories and a comparison with the measured data can be found in [3], and a description of the experiment itself in [2].

ACKNOWLEDGEMENTS

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