INTRINSIC LIMIT OF FIELD HOMOGENEITY OF HELICAL DIPOLES

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Abstract

The intrinsic limit of field homogeneity and the relation between the field homogeneity and multipoles of helical dipoles are analytically and geometrically investigated. The field homogeneity of helical dipoles, deformed by the twisting of the 2D dipole is intrinsically limited by the degree of twist, differently from that of the 2D dipole. It results that the highest homogeneity of the dipole field for the circular homogeneous cross-sectional region is accomplished with the non-zero helical sextupole.

1 MAGNETIC FIELD OF IDEAL HELICAL DIPOLES

1.1 Interior magnetic field of helical dipole

The interior magnetic field of helical dipole coil can be expressed as follows, on the European definition, [1 - 3]

$$\begin{cases} B_r(r,\theta,z) = B_{ref}(k)\, r_0 \sum_{n=1}^{\infty} n! \left[\dfrac{2}{n\,k\,r_0}\right]^n k\, I_n'(n\,k\,r) \times \\ \quad \left\{-a_n(k) \cos\left(n(\theta - k\,z)\right) + b_n(k) \sin\left(n(\theta - k\,z)\right)\right\} \\ B_\theta(r,\theta,z) = B_{ref}(k)\, r_0 \sum_{n=1}^{\infty} n! \left[\dfrac{2}{n\,k\,r_0}\right]^n \dfrac{I_n(n\,k\,r)}{r} \times \\ \quad \left\{a_n(k) \sin\left(n(\theta - k\,z)\right) + b_n(k) \cos\left(n(\theta - k\,z)\right)\right\} \\ B_z(r,\theta,z) = B_{ref}(k)\, r_0 \sum_{n=1}^{\infty} (-\,k)\, n! \left[\dfrac{2}{n\,k\,r_0}\right]^n I_n(n\,k\,r) \times \\ \quad \left\{a_n(k) \sin\left(n(\theta - k\,z)\right) + b_n(k) \cos\left(n(\theta - k\,z)\right)\right\} \end{cases} \tag{1}$$

where $k = 2\pi/L$, L is the helical pitch length, r_0 is the reference radius, and $I_n(nkr)$ is the modified Bessel function of the first kind of order n, and $K_n(nkr)$ is the modified Bessel function of the second kind of order n. Therefore, the y component of field at z=0, $B_y(r,\theta,z=0)$ becomes,

$$B_y(r,\theta,z=0) = B_r(r,\theta,z=0) \sin\theta + B_\theta(r,\theta,z=0) \cos\theta \tag{2}$$

On the case with $a_n(k)=b_{2n}(k)=0$ for n=1, 2, 3, ... , ∞, corresponding to the dipole symmetry,

$$\begin{aligned} B_y(r,\theta,z=0) &= B_{ref}(k) \sum_{n=1,3,5}^{\infty} b_n(k)\, r_0\, n! \left[\frac{2}{n\,k\,r_0}\right]^n \times \\ &\left\{k\, I_n'(n\,k\,r) \sin n\theta \sin\theta + \frac{I_n(n\,k\,r)}{r} \cos n\theta \cos\theta\right\} \\ &= B_{ref}(k) \sum_{n=1,3,5}^{\infty} b_n(k)\, M_n(k,r,\theta) \end{aligned} \tag{3}$$

The limiting forms for small argument of $k\, I_n'(nkr)$ and $I_n(nkr)/r$ are as follows, [4]

$$\begin{cases} \lim_{k\to 0} \left[k\, I_n'(n\,k\,r)\right] \\ \quad = \lim_{k\to 0} \left[k \left(I_{n-1}(n\,k\,r) - \dfrac{1}{k\,r} I_n(n\,k\,r)\right)\right] \\ = k \left(\dfrac{1}{(n-1)!} \left(\dfrac{n\,k\,r}{2}\right)^{n-1} - \dfrac{1}{k\,r} \dfrac{1}{n!} \left(\dfrac{n\,k\,r}{2}\right)^n\right) = \dfrac{1}{r} \dfrac{1}{n!} \left(\dfrac{n\,k\,r}{2}\right)^n \\ \lim_{k\to 0} \left[\dfrac{I_n(n\,k\,r)}{r}\right] = \dfrac{1}{r} \dfrac{1}{n!} \left(\dfrac{n\,k\,r}{2}\right)^n \end{cases} \tag{4}$$

Then, the asymptotic form for the y component of field at z=0, $B_y(r,\theta,z=0)$ as $k\to 0$ ($L\to\infty$) becomes,

$$\begin{aligned} &\lim_{k\to 0} \left[B_{y,helix}(r,\theta,z=0)\right] = B_{y,2d}(r,\theta) \\ &\quad = B_{ref} \sum_{n=1,3,5}^{\infty} \left(\frac{r}{r_0}\right)^{n-1} b_n \cos\left[(n-1)\theta\right] \end{aligned} \tag{5}$$

Then, it results that the asymptotic form of B_y for helical dipoles is equal to the form for 2-dimensional dipoles.

1.2 About an ideal helical dipole

With the assumption of $b_1=1$, $b_3=0$, $b_5=0$, ... , corresponding to the ideal dipole, the y component of field at $r=r_0$, z=0, $By(r=r_0,\theta,z=0)$ becomes,

$$\begin{aligned} &B_y(r=r_0,\theta,z=0)\big|_{n=1} = B_{ref}(k)\, M_1(k,r_0,\theta) \\ &\quad = B_{ref}(k) \frac{2}{k} \left\{k\, I_1'(k\,r_0) \sin^2\theta + \frac{I_1(k\,r_0)}{r_0} \cos^2\theta\right\} \\ &= B_{ref}(k) \left\{I_0(k\,r_0) + \left(-\,I_0(k\,r_0) + \frac{2}{k} \frac{I_1(k\,r_0)}{r_0}\right) \cos 2\theta\right\} \\ &\quad = B_{ref}(k) \left\{1.00165 - 8.24 \times 10^{-4} \cos 2\theta\right\} \end{aligned} \tag{6}$$

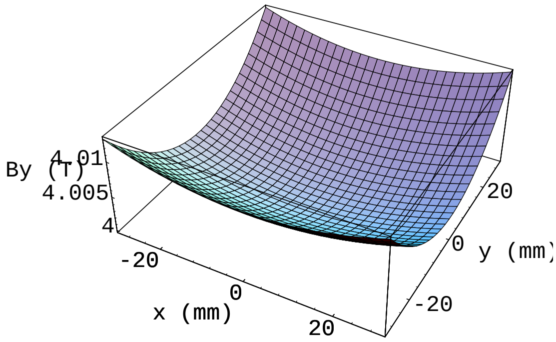

Fig. 1 3D plot of B_y for the helical dipole, with b_1=1, and b_3, b_5, ...=0.

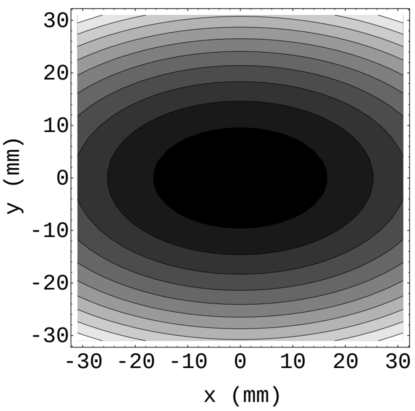

Fig. 2 Contour plot of B_y for the helical dipole, with b_1=1, and b_3, b_5, ...=0.

where the helical pitch length L=2.4 m, $k = 2\pi/2.4 = 2.62$, r_0=31 mm are assumed. The 3D and contour plot of B_y for the ideal dipole are shown in Figs. 1 and 2, with $B_{ref}(k) = B_y(r=0,\theta,z=0) = B_{y0} = 4.0$ T. Similarly, the sextupole term of field at $r=r_0$, z=0, $By(r=r_0,\theta,z=0)$ becomes,

$$\begin{aligned} B_y(r=r_0,\theta,z=0)\big|_{n=3} &= B_{ref}(k)\, b_3(k)\, M_3(k,r_0,\theta) \\ &= B_{ref}(k)\, b_3(k)\, r_0\, 3!\left[\frac{2}{3\,k\,r_0}\right]^3 \times \\ &\left\{\frac{k\, I_2(3\,k\,r_0)}{2}\cos 2\theta + \left(-\frac{k\, I_2(3\,k\,r_0)}{2} + \frac{I_3(3\,k\,r_0)}{r_0}\right)\cos 4\theta\right\} \\ &= B_{ref}(k)\, b_3(k)\left\{1.00495\cos 2\theta - 1.24\times 10^{-3}\cos 4\theta\right\} \end{aligned} \quad (7)$$

Therefore, with the following value of the helical sextupole coefficient, $b_3(k)$,

$$b_3(k) = -\frac{\left(-I_0(k\,r_0) + \dfrac{2}{k}\dfrac{I_1(k\,r_0)}{r_0}\right)}{r_0\, 3!\left[\dfrac{2}{3kr_0}\right]^3 \dfrac{k\, I_2(3\,k\,r_0)}{2}} \approx 8.20\times 10^{-4} \quad (8)$$

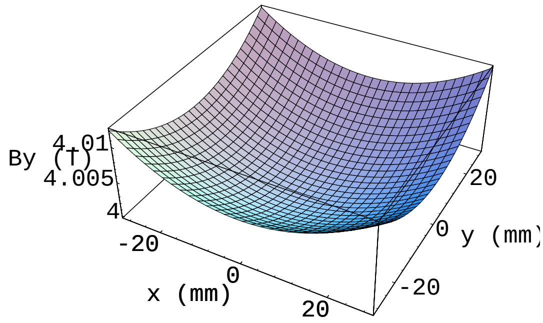

Fig. 3 3D plot of B_y for the helical dipole with b_1=1, b_3=0.00082, and b_5, b_7, ...=0.

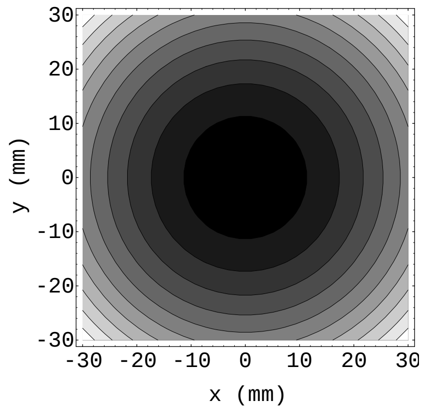

Fig. 4 Contour plot of B_y for the helical dipole, with b_1=1, b_3=0.00082, and b_5, b_7, ...=0.

the cos 2θ term of $B_y(r=r_0,\theta,z=0)$ vanishes. As a result, with the assumption of $b_1(k) = 1$, $b_3(k) = 8.2\times 10^{-4}$, $b_5(k) = 0$, ... , corresponding to the modified ideal dipole, the y component of field at $r=r_0$, z=0, $By(r=r_0,\theta,z=0)$ becomes,

$$\begin{aligned} B_y(r=r_0,\theta,z=0) &= B_{ref}(k)\left\{M_1(k,r_0,\theta) + M_3(k,r_0,\theta)\right\} \\ &= B_{ref}(k)\left\{1.00165 - 1.015\times 10^{-6}\cos 4\theta\right\} \end{aligned} \quad (9)$$

The 3D and contour plot of B_y for this modified ideal dipole, with $b_1(k) = 1$, $b_3(k) = 8.2\times 10^{-4}$, $b_5(k) = 0$, ... , are shown in Figs. 3 and 4. Therefore, it can be recognized that the homogeneity of the dipole field B_y at the circular region of r=31 mm is limited. As a result, with the helical sextupole coefficient $b_3(k) = 8.2\times 10^{-4}$, the minimum value of $|(B_y(r,\theta,z=0) - B_{y0})/B_{y0}|$ is about 0.165 % which is intrinsically determined from the value of the modified Bessel function of the first kind of order 0, $I_0(kr_0)$, depending on the twist or k.

2 RELATION BETWEEN FIELD HOMOGENEITY AND HELICAL MULTIPOLES

The relation between the field homogeneity at the circular region of r=31 mm and helical multipoles can be also geometrically investigated. The field homogeneity of the interior magnetic field for helical dipole coils can be

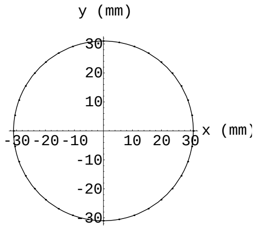

Fig. 5 Homogeneous region of B_y with a circular shape.

expressed as follows with the definition of $b_1(k)=1$, differently from that of the 2D dipole,

$$\left|\frac{B_y(k,r,\theta,z=0) - B_{ref}(k)}{B_{ref}(k)}\right|$$
$$= \left|(M_1(k,r,\theta) - 1) + b_3(k)\ M_3(k,r,\theta) + b_5(k)\ M_5(k,r,\theta) + \ldots\right| \tag{10}$$

Therefore, the requirement condition for the helical multipoles can be calculated from the prescribed homogeneity of the dipole field B_y at the boundary points of the circular region of r=31 mm shown in Fig. 5, using Eq.(10). For example, for $|(B_y(r,\theta,z=0) - B_{y0})/B_{y0}|$ of 0.2 % and 0.4 %, with $b_7(k)=0$, $b_9(k)=0,\cdots$, the satisfying region of $(b_3(k), b_5(k))$ for the prescribed field difference at (r=31 mm, θ=0) is shown in Fig. 6. The darker region in Fig. 6, corresponds to $|(B_y(r,\theta,z=0) - B_{y0})/B_{y0}|$ of 0.2 %. The resultant satisfying regions of $(b_3(k), b_5(k))$ for all boundary points for $|(B_y(r,\theta,z=0) - B_{y0})/B_{y0}|$ of 0.18 % and 0.3 % are shown as the central white zones in Figs. 7 and 8, respectively. Therefore, it is recognized that the satisfying region of $(b_3(k), b_5(k))$ with $b_7(k)=0$, $b_9(k)=0,\cdots$, for all boundary points vanishes for $|(B_y(r,\theta,z=0) - B_{y0})/B_{y0}|$ of < 0.165 %. This result is equivalent with Eq.(9).

3 CONCLUSION

The intrinsic limit of field homogeneity and the relation between field homogeneity and helical multipoles of helical dipoles are obtained. It can be realized that the field homogeneity of helical dipoles is significantly different from that of the ordinary 2D dipoles. This relation will be useful to optimize the cross-sectional shape of helical dipole coils.

In addition, this studies will be applicable for the estimation of the effect of geometrical distortions of the ordinary 2D dipole due to the twist.

4 REFERENCES

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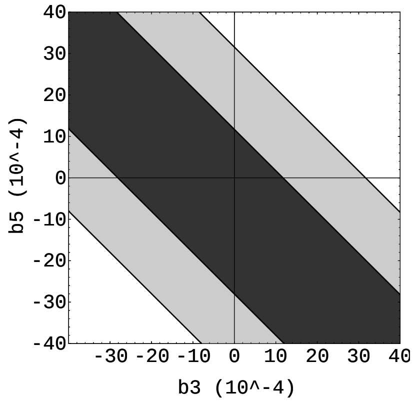

Fig. 6 (b_3, b_5) relation of < 0.2 % and < 0.4 % of $|B_y-B_{y0}|/B_{y0}$ at (x = 31 mm, y = 0 mm) with b_7=0, and b_9, b_{11}, ...=0.

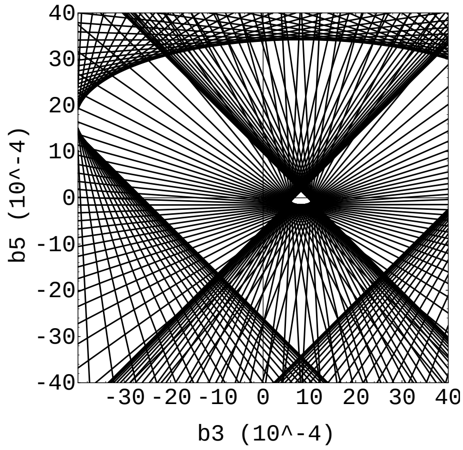

Fig. 7 (b_3, b_5) relation of < 0.18 % of $|B_y-B_{y0}|/B_{y0}$ with b_7=0, and b_9, b_{11}, ...=0.

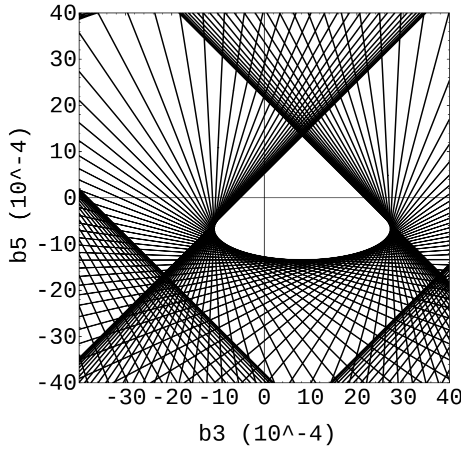

Fig. 8 (b_3, b_5) relation of < 0.3 % of $|B_y-B_{y0}|/B_{y0}$ with b_7=0, and b_9, b_{11}, ...=0.

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