Theoretical study of transverse-longitudinal emittance coupling

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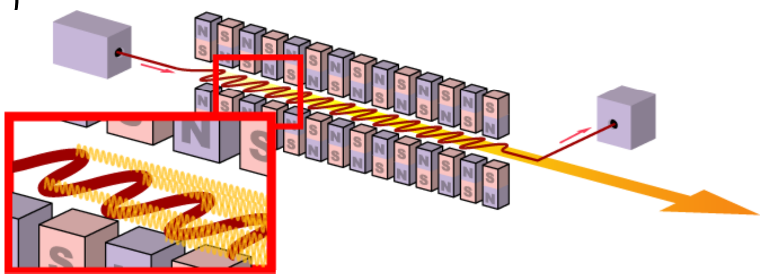
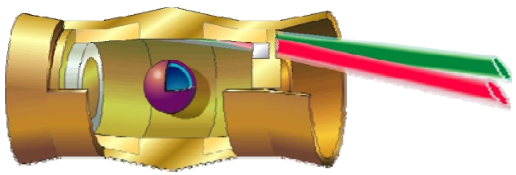
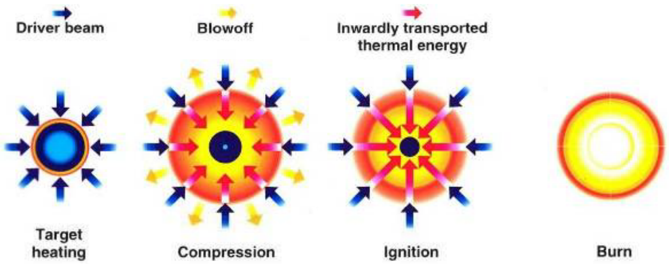
Transverse-longitudinal emittance coupling gives better beams

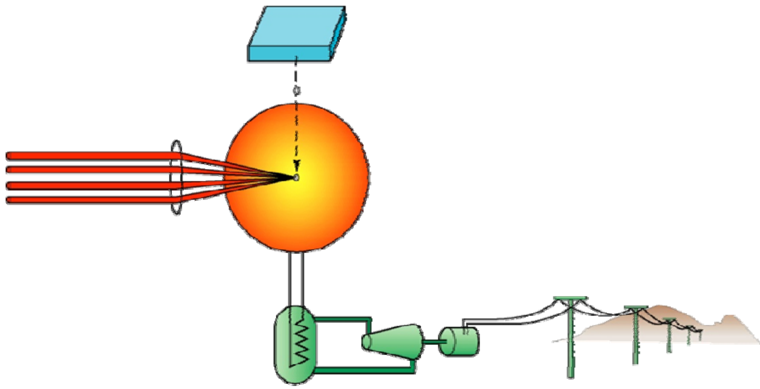
- FEL (LCLS, Emma 06', Kim 03')



- Heavy ion fusion







Emittance dynamics from covariance matrix

- Two issues with coupled emittance dynamics

$$\sigma \equiv \begin{pmatrix} \langle x^2 \rangle & \langle xz \rangle & \langle xx' \rangle & \langle xz' \rangle \\ \langle xz \rangle & \langle z^2 \rangle & \langle zx' \rangle & \langle zz' \rangle \\ \langle x'x \rangle & \langle x'z \rangle & \langle x'^2 \rangle & \langle x'z' \rangle \\ \langle z'x \rangle & \langle z'z \rangle & \langle x'z' \rangle & \langle z'^2 \rangle \end{pmatrix} \qquad \langle \ \rangle \equiv \int f_b \, dx\, dz\, dp_x\, dp_z$$

$$\sigma(s) = M(s)^T \sigma_0 M(s)$$

Transfer matrix for coupled dynamics

Initial covariance matrix

Previous work and present study

- Emittance exchange in one pass through the coupling component.
  - Emma *et al* 06', Cornacchia *et al* 02', Kim 03'
- Eigen-emittance
  - Williamson's theorem 36'.
  - Dragt's book, Yampolsky *et al* 11'.
  - Kishek *et al* 99'.

- How about the transfer matrix *M(s)* ?
- How about a coupled lattice?
  - Emittance exchange?

} Present study

2D coupled transverse dynamics

What is $M(t)$?

--- $M(t) \in Sp(4,\mathbb{R})$

10 free parameters

$$H = \frac{1}{2} q^T A q, \quad q = (x, z, \dot{x}, \dot{z})^T$$

$$A = \begin{pmatrix} \kappa & 0 \\ 0 & I \end{pmatrix}, \quad \kappa = \begin{pmatrix} \kappa_x & \kappa_{xz} \\ \kappa_{xz} & \kappa_z \end{pmatrix}$$

skew-quadrupole

Transfer matrix

Original Courant-Snyder theory
for uncoupled dynamics:

$SO(2)$

$$M(t) = \begin{pmatrix} w & 0 \\ \dot{w} & \frac{1}{w} \end{pmatrix} \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} w_0^{-1} & 0 \\ -\dot{w}_0 & w_0 \end{pmatrix}$$

scalar

Non-commutative generalization
for coupled dynamics:

$$M_c(t) = \begin{pmatrix} w^T & 0 \\ w^{-1}\dot{w}w^T & w^{-1} \end{pmatrix} \begin{pmatrix} P_1 & -P_2 \\ P_2 & P_1 \end{pmatrix} \begin{pmatrix} w_0^{-1T} & 0 \\ -\dot{w}_0 & w_0 \end{pmatrix}$$

2×2

$SO(4)$

Envelope equation

Original Courant-Snyder theory
for uncoupled dynamics:

Envelope scalar

$$w''(s) + \kappa(s) w(s) = w^{-3}(s)$$

Non-commutative generalization
for coupled dynamics:

$$w''(s) + w(s)\kappa(s) = w^{-1T} w^{-1} w^{-1T}$$

2×2 envelope matrix

Courant-Snyder Invariant

Original Courant-Snyder theory for uncoupled dynamics:

$$I = (q, \dot{q}) \begin{pmatrix} w^{-1} & -\dot{w} \\ 0 & w \end{pmatrix} \begin{pmatrix} w^{-1} & 0 \\ -\dot{w} & w \end{pmatrix} \begin{pmatrix} q \\ \dot{q} \end{pmatrix} = \frac{q^2}{w^2} + (w\dot{q} - \dot{w}q)^2$$

w: envelope scalar

Non-commutative generalization for coupled dynamics:

$$I = (x^T, \dot{x}^T) \begin{pmatrix} w^{-1} & -\dot{w}^T \\ 0 & w^T \end{pmatrix} \begin{pmatrix} w^{-1T} & 0 \\ -\dot{w} & w \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} = (x^T w^{-1} w^{-1T} x) + (\dot{x}^T w^T - x^T \dot{w}^T)(w\dot{x} - \dot{w}x)$$

w: 2×2 envelope matrix

Phase advance

Original Courant-Snyder theory for uncoupled dynamics:

$$\dot{\varphi} \equiv \begin{pmatrix} 0 & -w^{-2} \\ w^{-2} & 0 \end{pmatrix} \in so(2)$$

$$\dot{P} = P\dot{\varphi}$$

$$P = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \in SO(2)$$

Non-commutative generalization for coupled dynamics:

$$\dot{\varphi} \equiv \begin{pmatrix} 0 & -w^{-1T} w^{-1} \\ w^{-1T} w^{-1} & 0 \end{pmatrix} \in so(4)$$

$$\dot{P} = P\dot{\varphi}$$

$$P = \begin{pmatrix} P_1 & -P_2 \\ P_2 & P_1 \end{pmatrix} \in SO(4)$$

Phase advance

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Non-commutative generalization for coupled dynamics:

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$$\dot{P} = P\dot{\varphi}$$

$$P = \begin{pmatrix} P_1 & -P_2 \\ P_2 & P_1 \end{pmatrix} \in SO(4)$$

Twiss functions

Original Courant-Snyder theory for uncoupled dynamics:

$$\beta = w^2$$
$$\alpha = -ww'$$
$$\gamma = w^{-2} + w'^2$$

↓

Non-commutative generalization for coupled dynamics:

$$\beta = w^T w$$
$$\alpha = -w^T w'$$
$$\gamma = \left(w^T w\right)^{-1} + w'^T w'$$

How did we do it? General problem

$2n \times 2n$

$$H = \frac{1}{2} q^T A(t) q$$

$$q = (x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)^T$$

Hamiltonian Eq.

$$\dot{q} = J \nabla H, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

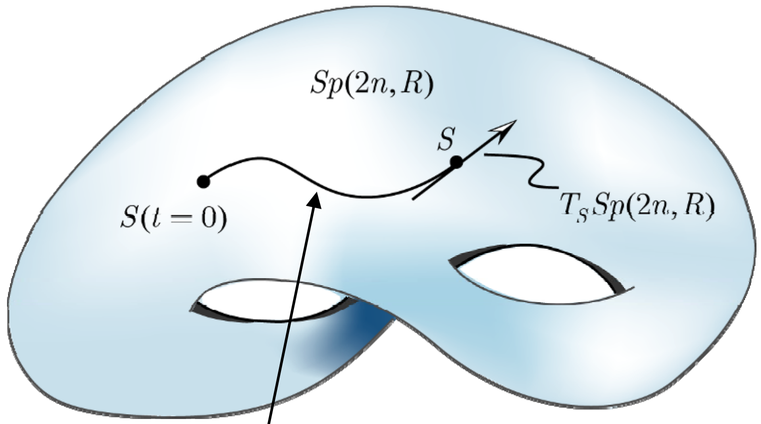
Time-dependent canonical transformation $S(t)$

$$\overline{q} = S(t)q$$

$$\overline{H} = \frac{1}{2}\overline{q}^T \overline{A}(t)\overline{q}$$

Target Hamiltonian

Symplectic group: $SJS^T = J$



$$\dot{S} = J\bar{A}S - SJA$$

Two transformations to $\overline{A}(t) = 0$

Dynamics of physical emittance

$$\sigma\left(s\right) = M(s)^T \sigma_0 M(s)$$

- Eignen-emittances are invariants.

- But physical emittance measures beam qualities

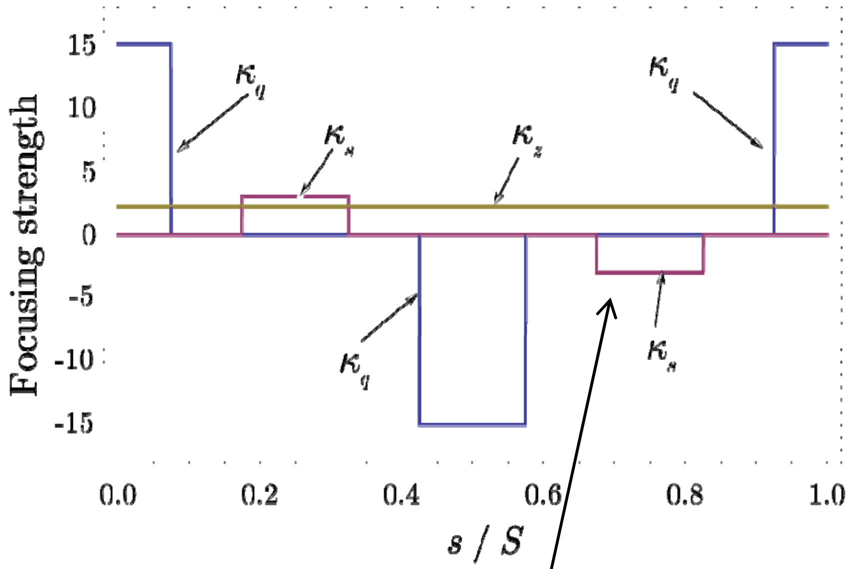
$$\varepsilon_{4D} = \sqrt{Det[\sigma]} = \varepsilon_{4D}(s=0)$$

$$\varepsilon_x^2 \equiv Det(\sigma_x),\ \sigma_x \equiv \begin{pmatrix} \left\langle x^2 \right\rangle & \left\langle xx' \right\rangle \\ \left\langle xx' \right\rangle & \left\langle x'^2 \right\rangle \end{pmatrix}$$

$$\varepsilon_z^2 \equiv Det(\sigma_z),\ \sigma_z \equiv \begin{pmatrix} \left\langle z^2 \right\rangle & \left\langle zz' \right\rangle \\ \left\langle zz' \right\rangle & \left\langle z'^2 \right\rangle \end{pmatrix}$$

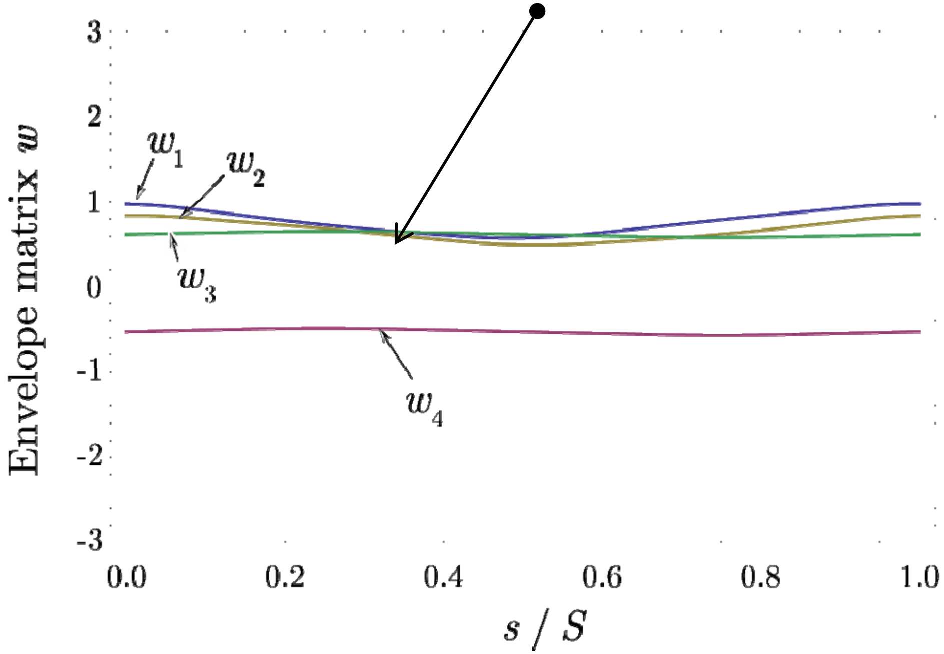
$$\varepsilon_{xz}^2 \equiv Det(\sigma_{xz}),\ \sigma_{xz} \equiv \begin{pmatrix} \left\langle x^2 \right\rangle & \left\langle xz \right\rangle \\ \left\langle xz \right\rangle & \left\langle z^2 \right\rangle \end{pmatrix}$$

Example

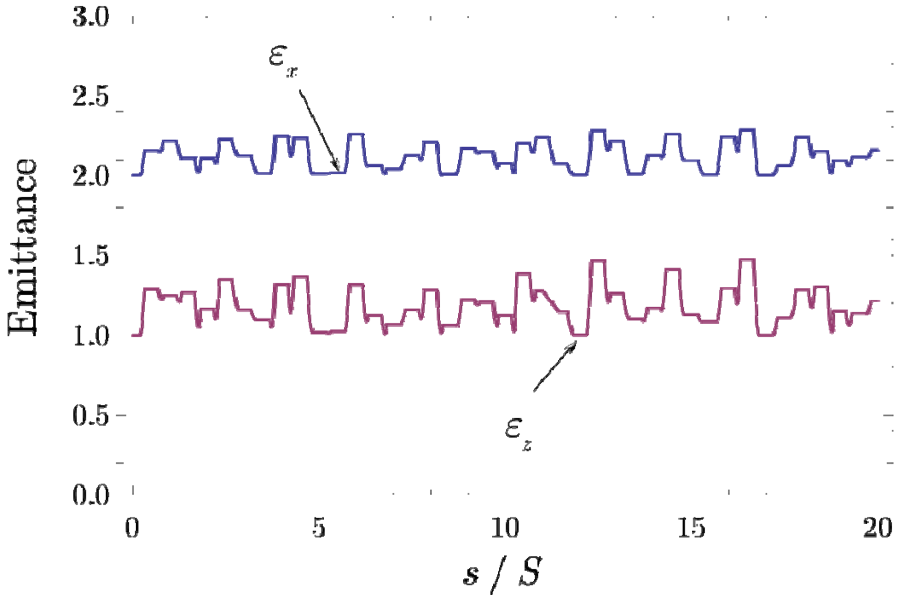


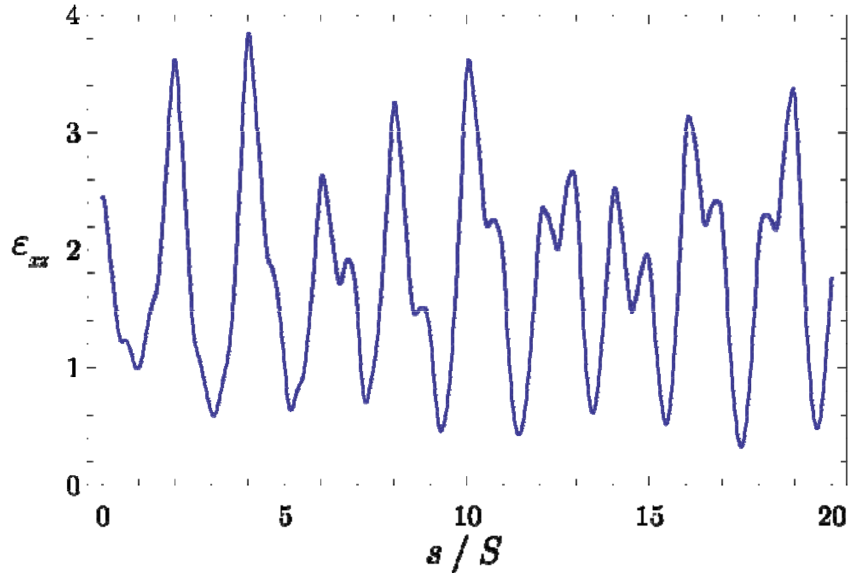
Coupling skew-quad

Matched solution of the envelope matrix w



Interesting dynamics of physical emittance





❑ Interesting dynamics

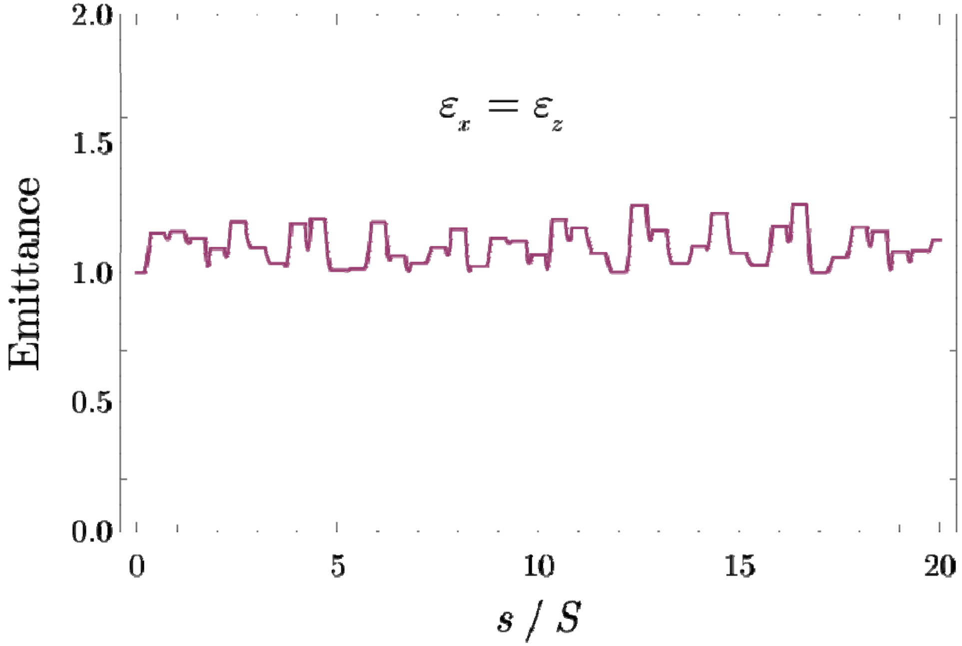
✓ Emittance does not match the lattice.

✓ No emittance exchange.

✓ Classical uncertain principle and minimum emittance theorem.

❑ But can be explored for beam better property at the focal point.

An interesting case



$$\varepsilon_{x0} = \varepsilon_{z0} \longmapsto \varepsilon_x = \varepsilon_z$$

But, $\varepsilon_x = \varepsilon_z \neq const.$

Conclusions

- Coupled focusing lattice generates interesting emittance dynamics.
- Generalized Courant-Snyder theory for coupled dynamics is the tool to understand the coupled emittance dynamics.
- Coupled emittance dynamics can be explored for better beam properties at the focal point.

Courant-Snyder theory for uncoupled dynamics

$$q''(s) + \kappa(s)q(s) = 0$$

Phase advance

$$\varphi(t) = \int_0^t \frac{dt}{w^2(t)}$$

Courant (1958)

$$q(s) = I w(s) \cos[\varphi(s) + \varphi_0]$$

CS invariant

envelope

$$\begin{pmatrix} q \\ \dot{q} \end{pmatrix} = M(t) \begin{pmatrix} q_0 \\ \dot{q}_0 \end{pmatrix} \qquad M(t) = \begin{pmatrix} \sqrt{\dfrac{\beta}{\beta_0}}[\cos\varphi + \alpha_0 \sin\varphi] & \sqrt{\beta\beta_0}\sin\varphi \\ -\dfrac{1+\alpha\alpha_0}{\sqrt{\beta\beta_0}}\sin\varphi + \dfrac{\alpha_0 - \alpha}{\sqrt{\beta\beta_0}}\cos\varphi & \sqrt{\dfrac{\beta_0}{\beta}}[\cos\varphi - \alpha\sin\varphi] \end{pmatrix}$$

Courant-Snyder theory for uncoupled dynamics

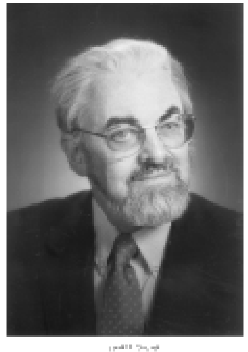
$$q''(s) + \kappa(s)q(s) = 0$$

Courant-Snyder invariant

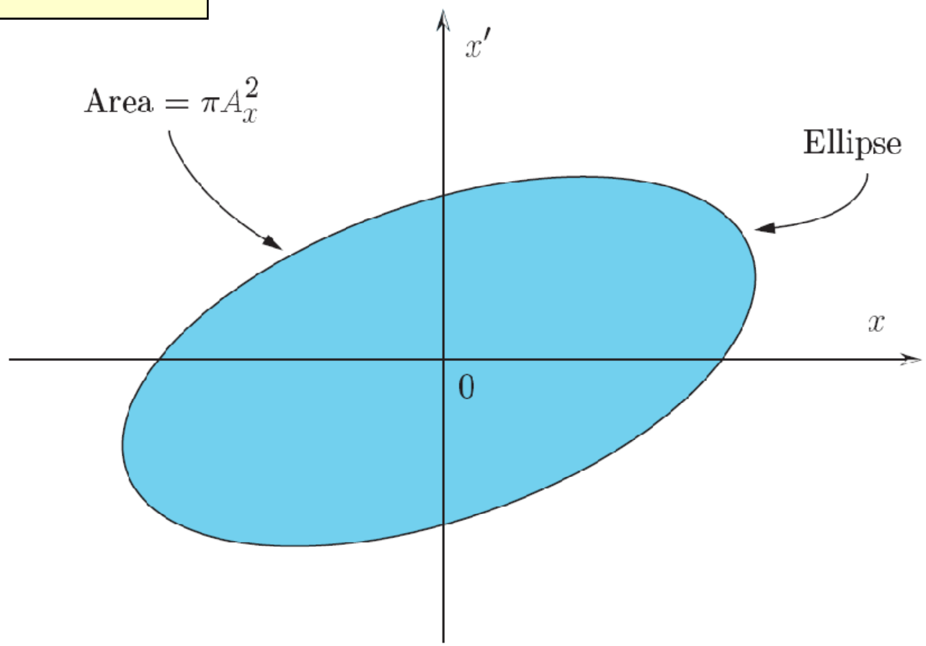
$$I^2 = \frac{q^2}{w^2} + \left(wq' - w'q\right)^2 = const.$$

$$w''(s) + \kappa(s)w(s) = w^{-3}(s)$$

Envelope eq.



Courant (1958)



Kapchinskij-Vladimirskij (KV) distribution (1959):

Uncoupled

The only known solution

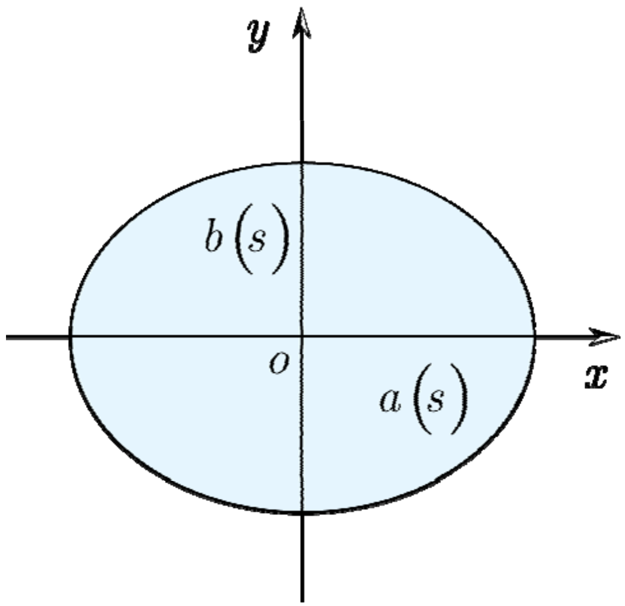
$$f_{KV} = \frac{N_b}{\pi^2 \varepsilon_x \varepsilon_y} \delta\left(\frac{I_x}{\varepsilon_x} + \frac{I_y}{\varepsilon_y} - 1\right),$$

$$I_x = \frac{x^2}{w_x^2} + \left(w_x \dot{x} - x \dot{w}_x\right)^2, \; I_y = \frac{y^2}{w_y^2} + \left(w_y \dot{y} - y \dot{w}_y\right)^2.$$

$$\ddot{w}_x + \kappa_x w_x = w_x^{-3}, \; \ddot{w}_y + \kappa_y w_y = w_y^{-3},$$

$$\kappa_x = \kappa_{qx} - \frac{2K_b}{a(a+b)}, \; \kappa_y = \kappa_{qy} - \frac{2K_b}{b(a+b)},$$

$$a \equiv \sqrt{\varepsilon_x} w_x, \; b \equiv \sqrt{\varepsilon_y} w_y.$$



Uncoupled

$$\kappa_{qxy} = \kappa_{qyx} = 0$$

Many ways [Teng, 71] to parameterize the transfer matrix


Lee Teng

Symplectic rotation form
[Edward-Teng, 73]:

$$Z = F^{-1} z$$

uncoupled

$$F = \begin{pmatrix} I\cos\phi & D^{-1}\sin\phi \\ -D\sin\phi & I\cos\phi \end{pmatrix}$$

No apparent
physical meaning

No β function

$$M(t) = F \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} F^{-1}$$

uncoupled CS
transfer matrix

Have to define beta function from particle trajectories?
[Ripken, 70], [Wiedemann, 99]

A hint from 1D C-S theory

$$M(t)=\begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}[\cos\varphi+\alpha_0\sin\varphi] & \sqrt{\beta\beta_0}\sin\varphi \\ -\frac{1+\alpha\alpha_0}{\sqrt{\beta\beta_0}}\sin\varphi+\frac{\alpha_0-\alpha}{\sqrt{\beta\beta_0}}\cos\varphi & \sqrt{\frac{\beta_0}{\beta}}[\cos\varphi-\alpha\sin\varphi] \end{pmatrix}$$

$$\beta(t)=w^2(t),\ \alpha(t)=-w\dot{w},\ \varphi(t)=\int_0^t\frac{dt}{\beta(t)}.$$

$$M(t)=\begin{pmatrix} w & 0 \\ \dot{w} & \frac{1}{w} \end{pmatrix}\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} w_0^{-1} & 0 \\ -\dot{w}_0 & w_0 \end{pmatrix}$$

Application: Strongly coupled system
Stability completely determined by phase advance

Theorem 1:

Unstable $\Leftrightarrow P_c(t)$ has an eigenvalue with $|\lambda| \neq 1$

one turn generalized phase advance

Theorem 2:

Stable $\Leftarrow J[P_c^T(t) - P_c(t)]$ is positive (negative)-definite

Application: Weakly coupled system
Stability determined by uncoupled phase advance

Theorem 1:

Unstable $\Leftrightarrow$ $P_c(t)$ has an eigenvalue with $|\lambda| \neq 1$

↓

Unstable $\Rightarrow$ uncoupled tune $\cos\phi_x = \cos\phi_y$

↕

uncoupled tune $\nu_x \pm \nu_y = n$

(sum and difference resonace)

Application: Weakly coupled system
Stability determined by uncoupled phase advance

Theorem 2:

Stable $\Leftarrow J[P_c^T(t) - P_c(t)]$ is positive (negative)-definite

$$\downarrow$$

$$\text{stable} \Leftarrow \sin\phi_x = \sin\phi_y$$

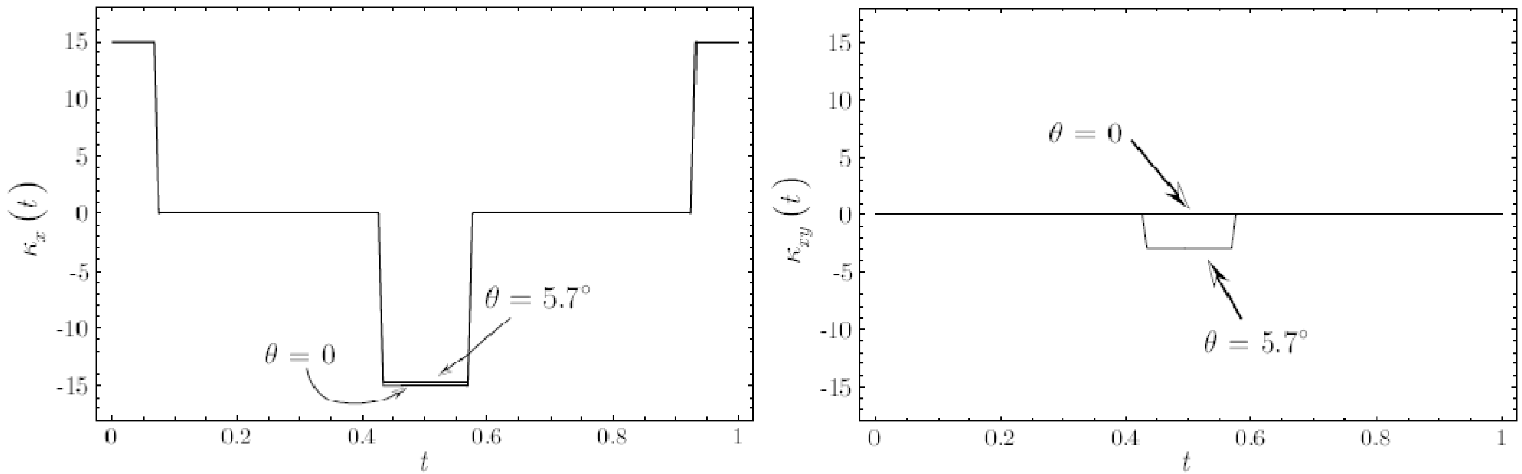
$$\downarrow$$

$$\text{unstable} \Rightarrow \nu_x + \nu_y = n \text{ (sum resonace)}$$

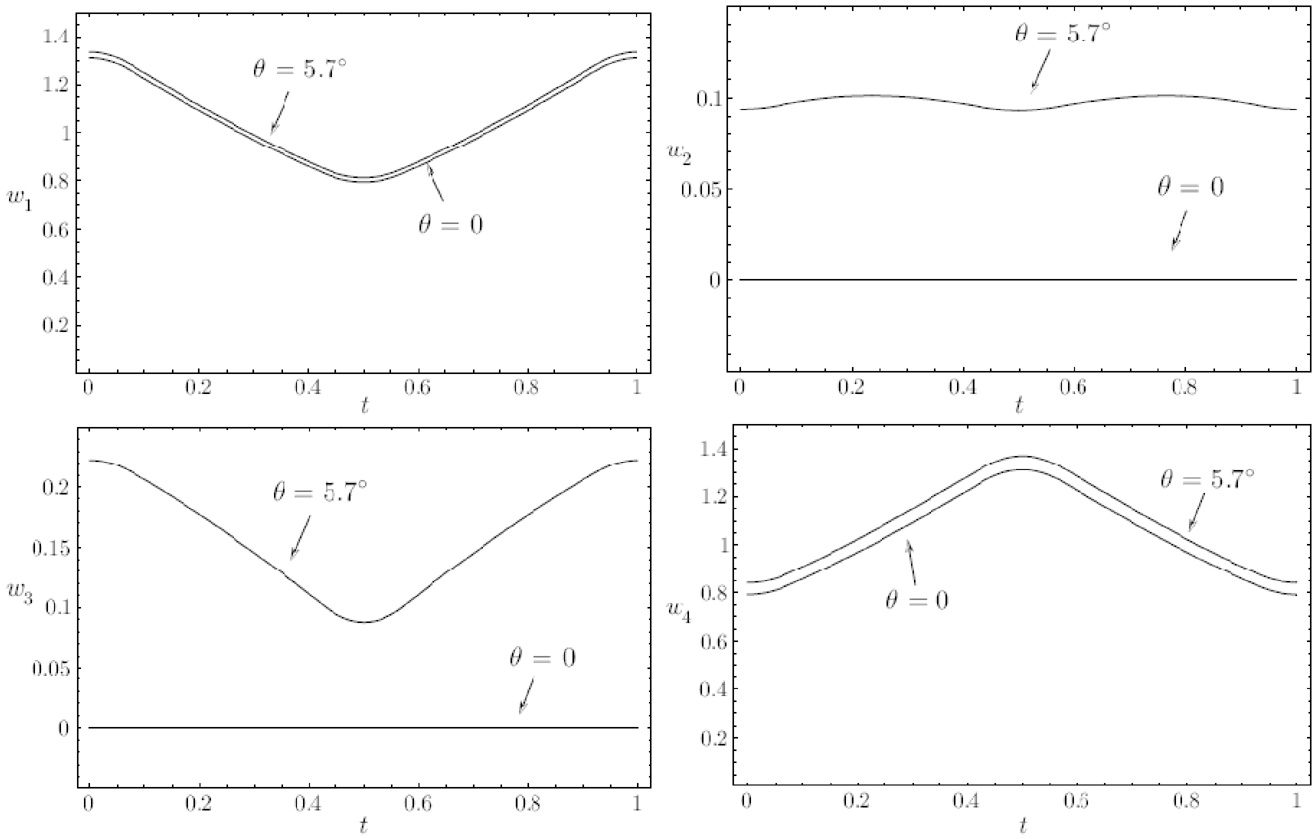
Numerical example – mis-aligned FODO lattice

mis-alignment angle

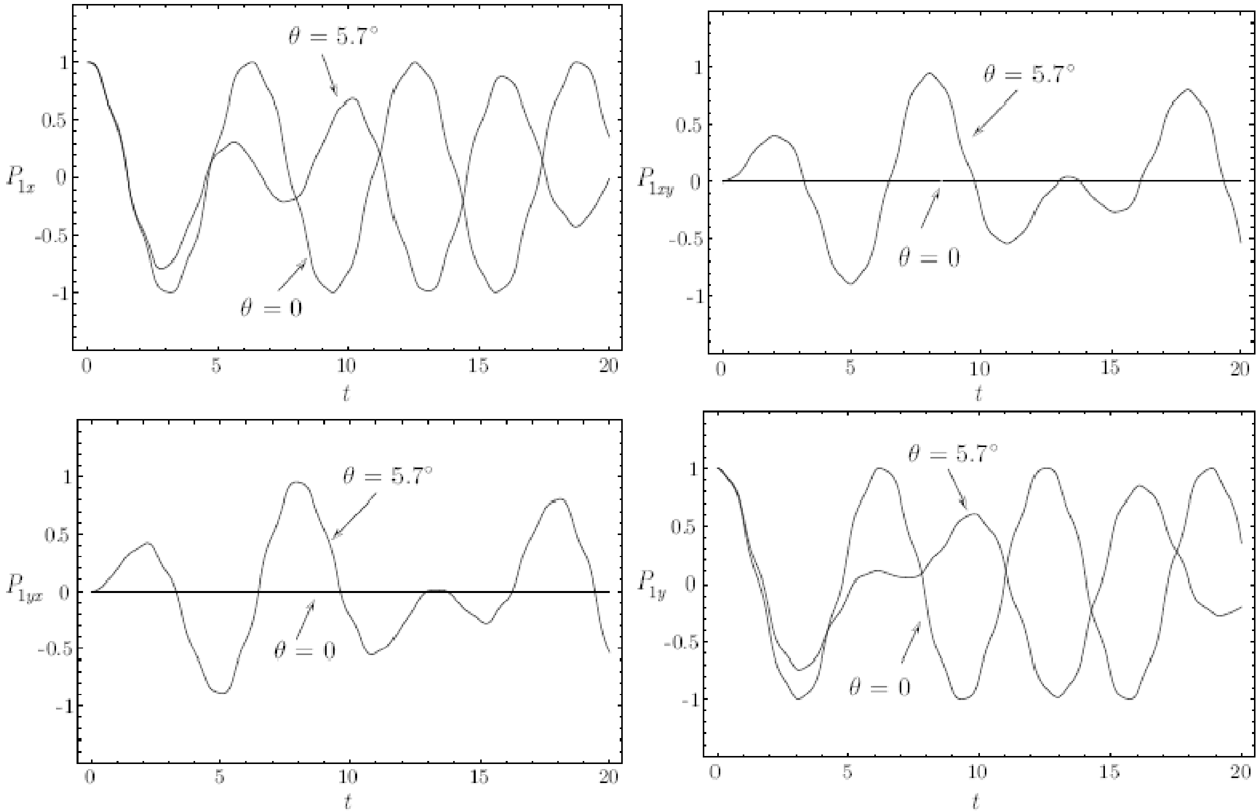
$$\kappa = \kappa_q \begin{pmatrix} \cos[2\theta(s)] & \sin[2\theta(s)] \\ \sin[2\theta(s)] & -\cos[2\theta(s)] \end{pmatrix} \qquad \text{[Barnard, 96]}$$



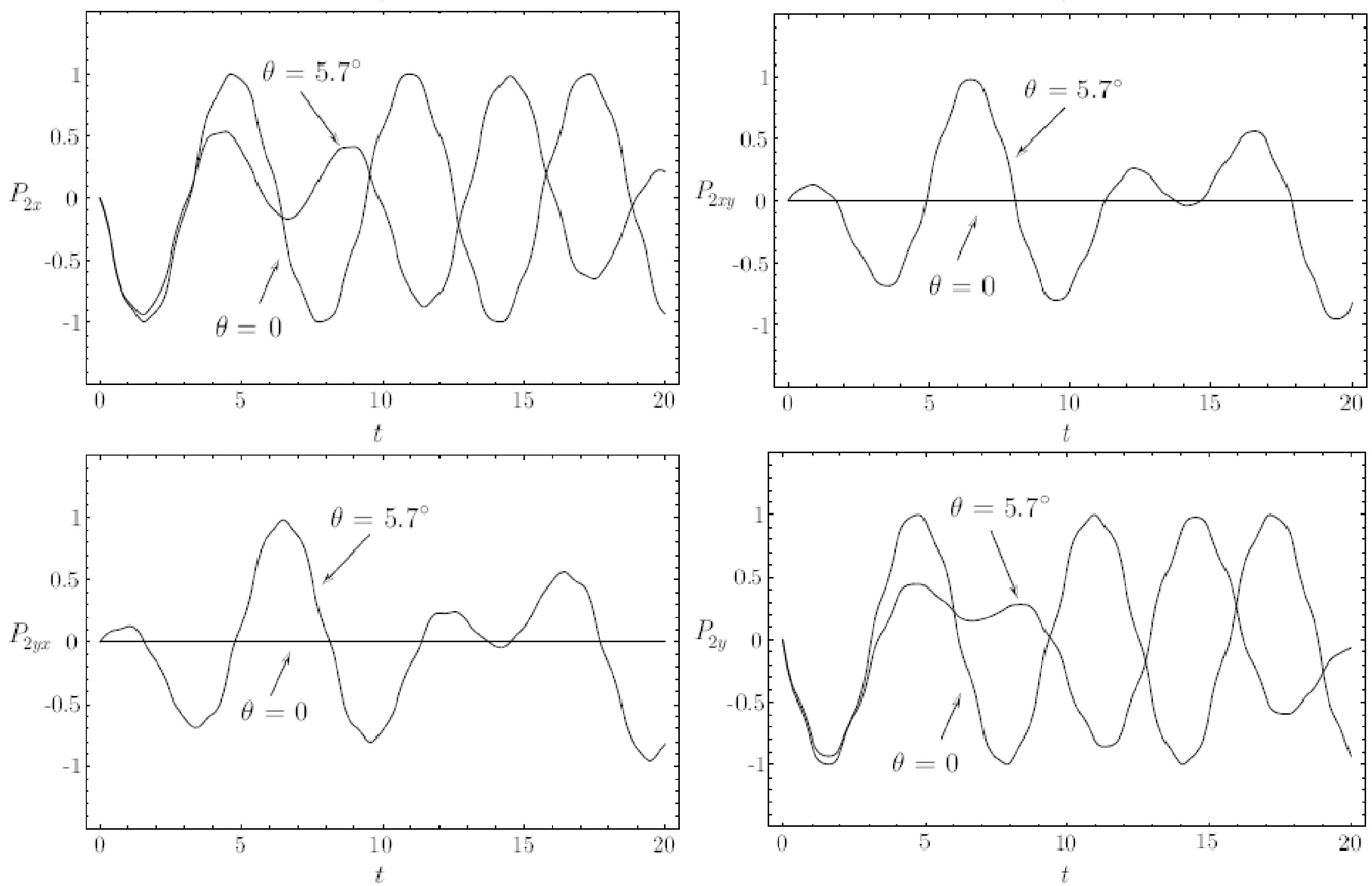
Envelope matrix w



Rotation matrix P_1



Rotation matrix P_2



Non-Commutative Courant-Snyder theory for coupled transverse dynamics

Step I

$$H_c = \frac{1}{2} z^T A_c z, \quad z = (x, y, \dot{x}, \dot{y})^T$$

$$A_c = \begin{pmatrix} \kappa & 0 \\ 0 & I \end{pmatrix}, \quad \kappa = \begin{pmatrix} \kappa_x & \kappa_{xy} \\ \kappa_{xy} & \kappa_y \end{pmatrix}$$

$$\overline{H}_c = \frac{1}{2} \overline{z}^T \overline{A}_c \overline{z}, \quad \overline{A}_c = \begin{pmatrix} \beta_I & 0 \\ 0 & \beta_I \end{pmatrix}$$

$$\dot{S} = J\overline{A}_c S - SJA_c$$

$$\begin{pmatrix} \dot{S}_1 & \dot{S}_2 \\ \dot{S}_3 & \dot{S}_4 \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \beta_I & 0 \\ 0 & \beta_I \end{pmatrix} \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} - \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \kappa & 0 \\ 0 & I \end{pmatrix}$$

$$S_2 = 0$$

$$\ddot{S}_4 = \beta_I^2 S_4 - S_4 \kappa$$

$w \equiv S_4$

$\beta \equiv \beta_I^{-1}$

$$\ddot{w} = \beta^{-2} w - w\kappa$$

S is symplectic:

$\beta^{-1} w w^T = I$

Non-Abelian envelope Eq.

$$\ddot{w} + w\kappa = \left(w^{-1}\right)^T w^{-1} \left(w^{-1}\right)^T$$

2×2

$$\overline{z} = Sz$$

$$S = \begin{pmatrix} \left(w^{-1}\right)^T & 0 \\ -\dot{w} & w \end{pmatrix}$$

2×2

$$S^{-1} = \begin{pmatrix} w^T & 0 \\ w^{-1}\dot{w}w^T & w^{-1} \end{pmatrix}$$

Step II

$$\bar{H}_c = \frac{1}{2}\bar{z}^T \bar{A}_c \bar{z}, \quad \bar{A}_c = \begin{pmatrix} w^{-1} & 0 \\ 0 & w^{-1} \end{pmatrix}$$

$$\bar{\bar{z}} = P\bar{z}$$

$$\bar{\bar{H}}_c = 0$$

$$\dot{P} = -PJ\bar{A}_c$$

$$P = \begin{pmatrix} P_1 & P_2 \\ -P_2 & P_1 \end{pmatrix} \in SO(4)$$

$$\dot{P} = P\dot{\varphi}$$

$$\dot{\varphi} \equiv \begin{pmatrix} 0 & -w^{-1T}w^{-1} \\ w^{-1T}w^{-1} & 0 \end{pmatrix} \in so(4)$$

Generalized KV distribution in coupled focusing lattice

$$\kappa_{qxy} = \kappa_{qyx} \neq 0$$

$$-\nabla\psi = -\kappa_s \mathbf{x},\ \kappa_s = \begin{pmatrix} \kappa_{sx} & \kappa_{sxy} \\ \kappa_{syx} & \kappa_{sy} \end{pmatrix}$$

$$-\nabla\psi - \kappa_q \mathbf{x} = -\kappa \mathbf{x},\ \kappa = \kappa_q + \kappa_s$$

$$\ddot{w} + w\kappa = \left(w^{-1}\right)^T w^{-1} \left(w^{-1}\right)^T$$

$$I_{CS} = \mathbf{x}^T w^{-1} w^{-1T} \mathbf{x} + \left(\dot{\mathbf{x}}^T w^T - \mathbf{x}^T \dot{w}^T\right)\left(w\dot{\mathbf{x}} - \dot{w}\mathbf{x}\right)$$

$$f_{KV} = \frac{N_b\,|\,w\,|}{A\varepsilon\pi}\delta\left(\frac{I_{CS}}{\varepsilon} - 1\right)$$

$$n(x,y,s) = \int d\dot{x}d\dot{y} f_{KV} = \begin{cases} N_b\,/\,A,\ 0 \le \mathbf{x}^T w^{-1} w^{-1T} \mathbf{x} < \varepsilon, \\ 0,\ \varepsilon < \mathbf{x}^T w^{-1} w^{-1T} \mathbf{x}. \end{cases}$$

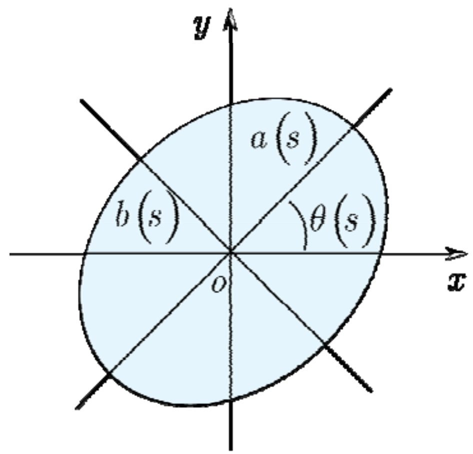
Envelope matrix equation

$$\ddot{w} + w\kappa = \left(w^{-1}\right)^T w^{-1} \left(w^{-1}\right)^T$$

$$\kappa = \kappa_q + \kappa_s$$

$$\kappa_s = \begin{pmatrix} \kappa_{sx} & \kappa_{sxy} \\ \kappa_{syx} & \kappa_{sy} \end{pmatrix}$$

$$\kappa_s = \frac{-2K_b}{a+b} Q \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix} Q^{-1}$$



Coupled

$$\beta^* \equiv w^{-1} w^{-1T}$$

$$Q^{-1} \beta^* Q = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

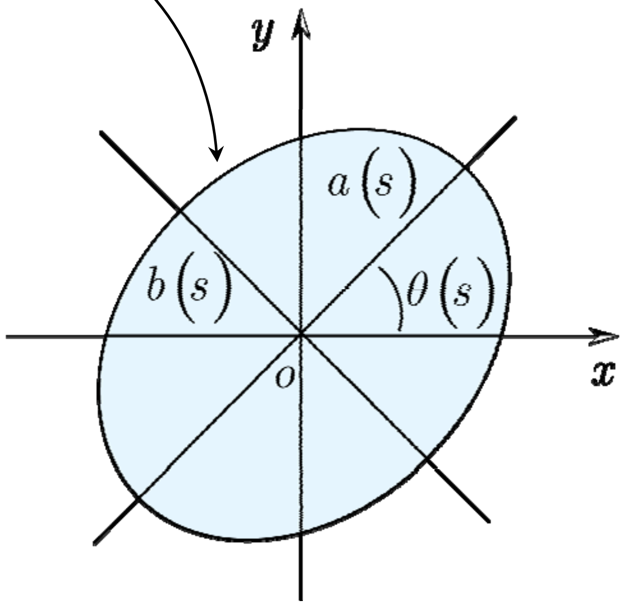
$$Q = \left(v_1, v_2\right)$$

$$a = \sqrt{\varepsilon / \lambda_1} \qquad b = \sqrt{\varepsilon / \lambda_2}$$

$$\mathbf{x}^T \beta^* \mathbf{x} = \varepsilon, \ \beta^* \equiv w^{-1} w^{-1T}$$

$$Q^{-1} \beta^* Q = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$Q = (v_1, v_2)$$



Coupled

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = 1$$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = Q^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$a = \sqrt{\varepsilon / \lambda_1}$$

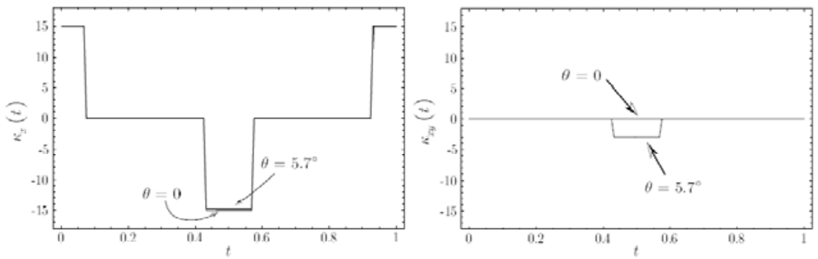
$$b = \sqrt{\varepsilon / \lambda_2}$$

$$-\begin{pmatrix} \partial\psi / \partial X \\ \partial\psi / \partial Y \end{pmatrix} = \frac{2K_b}{a+b} \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$-\begin{pmatrix} \partial\psi / \partial x \\ \partial\psi / \partial y \end{pmatrix} = -\kappa_s \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\kappa_s = \frac{-2K_b}{a+b} Q \begin{pmatrix} 1/a & 0 \\ 0 & 1/b \end{pmatrix} Q^{-1}$$

Generalized KV beam: pulsating & tumbling



$$\kappa = \kappa_q \begin{pmatrix} \cos[2\theta(s)] & \sin[2\theta(s)] \\ \sin[2\theta(s)] & -\cos[2\theta(s)] \end{pmatrix}$$

Skew quad rotation angle