Numerical and analytical studies of matched kinetic quasi-equilibrium solutions for an intense charged particle beam propagating through a periodic focusing quadrupole lattice.*

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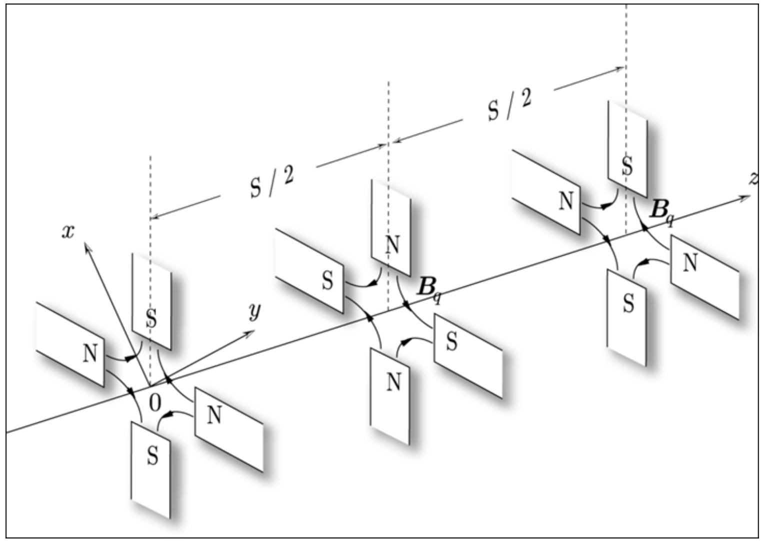
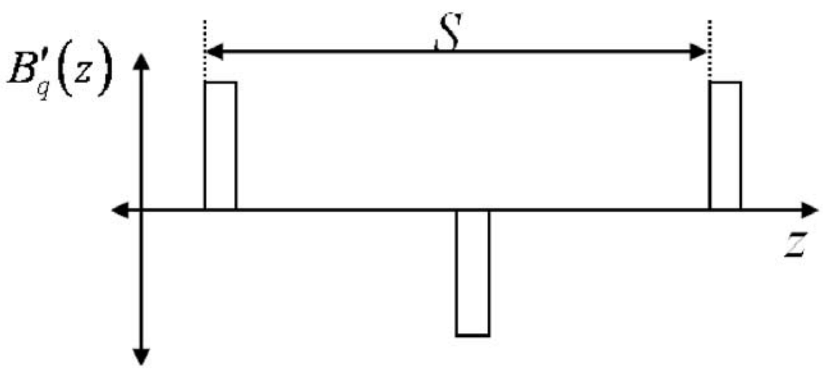
Alternating-Gradient Transport Systems Use a Spatially Periodic Lattice of Quadrupole Magnets for Transverse Confinement





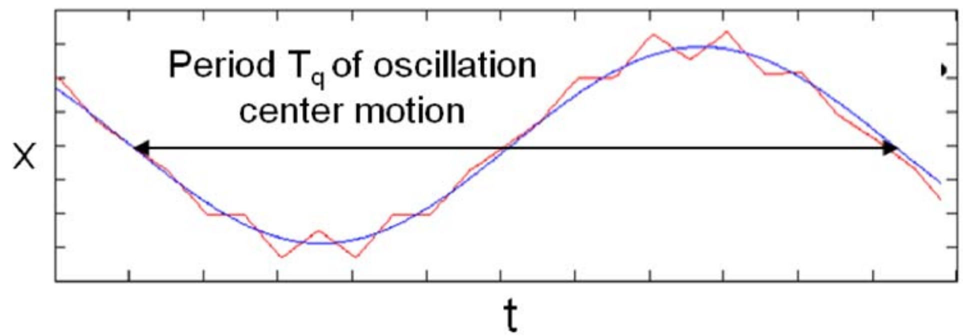
$$\boldsymbol{B}_q^{foc}(\boldsymbol{x}) = B_q'(z)\left(y\hat{\boldsymbol{e}}_x + x\hat{\boldsymbol{e}}_y\right)$$

$$\boldsymbol{F}_{foc}(\boldsymbol{x}) = -\kappa_q(z)\left(x\hat{\boldsymbol{e}}_x - y\hat{\boldsymbol{e}}_y\right)$$

$$\kappa_q(z) \equiv \frac{ZeB_q'(z)}{\gamma m \beta c^2}$$

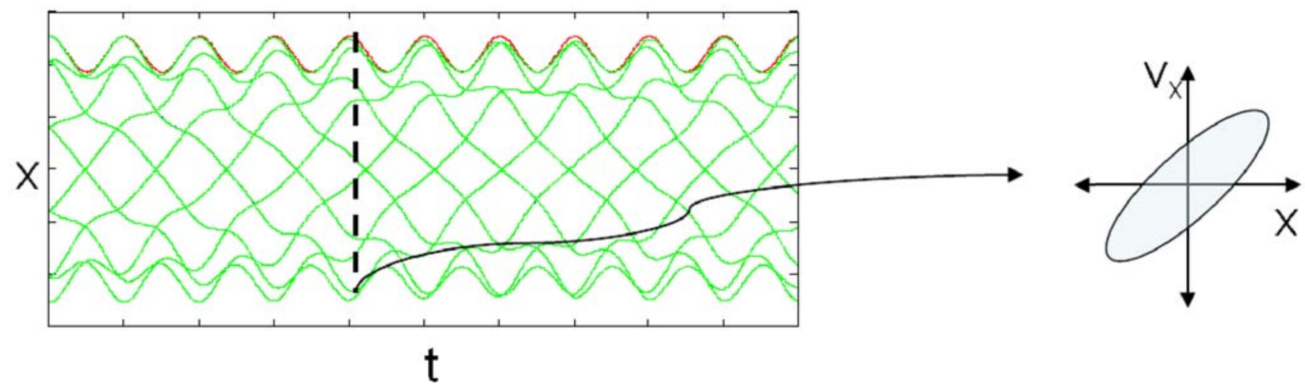
Particle trajectories are fast oscillations on top of slow motion of guiding centers

$\kappa_q(z)$: FODO lattice



$$\sigma_V = \omega_q / f$$

$\kappa_q(z)$: sinusoidal lattice



Motivations

- Identifying regimes for quiescent propagation of intense beams over long distances is of the great interest in accelerator research.
- In particular, the development of systematic theoretical approaches that are able to treat self-consistently the applied oscillating force and the nonlinear self-field force of the beam particles simultaneously has been a major challenge of modern beam physics.
- To determine matched-beam quasi-equilibrium distribution functions one needs to determine a dynamical invariant for the beam particles moving in the combined applied and self-generated fields.
- Typically, it is advantageous to eliminate fast oscillations from formalism and describe complex beam particle motion is in a new non-oscillating coordinates.
- Standard Hamiltonian techniques are cumbersome due to use of mixed oscillating and non-oscillating independent variables.

Abstract

- New Hamiltonian averaging technique has been developed, which incorporated both the applied periodic focusing force and the self-field force of the beam particles.

- Newly developed technique is specially designed to avoid use of oscillating independent variables. The method is analogous to the Lie transform methods in using only non-oscillating independent variables. At the same time the new approach retains the advantages of simplicity of Hamiltonian methods.

- Using the particle's vacuum phase advance $\epsilon = \sigma_v/2\pi$ treated as a small parameter, the perturbative series are constructed to transform away the fast particle orbit oscillations and obtain the average Hamiltonian accurate to order ϵ^3.

- The average Hamiltonian is an approximate invariant of the original system, and can be used to determine self-consistent beam quasi-equilibrium solutions that are matched to the focusing channel.

- Making use of this new method equations determining the average self-field potential for general boundary conditions has been obtained for the first time by taking into account the average contribution of the charges induced on the boundary.

Vlasov-Poisson system of equations

- The transverse dynamics of the intense charged particle beam can be described by the nonlinear Vlasov-Poisson system of equations for the beam distribution function $f(x, p, s)$ and the normalized self-field potential $\Psi(x, s)$.

- Here $s = v_b t$ is the longitudinal coordinate, and v_b is the directed beam velocity. The function $f(x, p, s)$ satisfies the nonlinear Vlasov equation

$$\frac{df}{ds} = \frac{\partial f}{\partial s} + \sum_{\alpha=1}^{2} \frac{dx^\alpha}{ds}\frac{\partial f}{\partial x^\alpha} + \sum_{\alpha=1}^{2} \frac{dp^\alpha}{ds}\frac{\partial f}{\partial p^\alpha} = 0,$$

- where the particle equations of motion are give by

$$\frac{dx^\alpha}{ds} = \frac{\partial H}{\partial p^\alpha}, \quad \frac{dp^\alpha}{ds} = -\frac{\partial H}{\partial x^\alpha},$$

- The Hamiltonian $H(x, p, s)$ describes the particle motion in a force field that is the sum of a linear, externally applied, transverse focusing force with components $F^\alpha_{foc} = -\kappa(s)\eta^\alpha x^\alpha$ and the normalized self-field potential $\Psi(x, s)$ is calculated self-consistently using Poisson's equation.

- $\kappa(s)$ is the focusing field strength $\eta^1 = 1$, $\eta^2 = -1$.

The Hamiltonian H for the particle motion

- It is convenient to introduce the re-normalized variables $\bar{x} = x/a$, $\bar{s} = s/S_0$, $\bar{\kappa}(s) = \kappa(s)/\kappa_0$, $\bar{p} = p/(a\kappa_0 S_0)$ and $\bar{f} = (f/N)a^4(\kappa_0 S_0)^2$, where S_0 is the characteristic period of the applied focusing force, a is the characteristic transverse beam dimension, and κ_0 is the characteristic value of the lattice function $\kappa(s)$.

- After the normalization, the Hamiltonian becomes

$$\bar{H}(\bar{x}, \bar{p}, \bar{s}) = \bar{\kappa}(\bar{s})\frac{[\eta^\alpha \bar{x}^\alpha \bar{x}^\alpha]}{2} + \epsilon\{\frac{[\bar{p}^\alpha \bar{p}^\alpha]}{2} + \int L(\bar{x}, \bar{x}')\bar{f}(\bar{x}', \bar{p}', s')D\bar{x}'D\bar{p}'\},$$

where $\epsilon \equiv S_0^2\kappa_0$ and $\int d\bar{x}d\bar{p}\bar{f} = 1$.

- We adopt the notation $[x^\alpha x^\alpha] \equiv \sum_{\alpha=1}^{2} x^\alpha x^\alpha$ and $\int dx_1 dx_2 Z = \int DxZ$.

- Green's function $L(\bar{x} - \bar{x}')$ satisfies the equation

$$\left[\frac{\partial}{\partial \bar{x}^\alpha}\frac{\partial}{\partial \bar{x}^\alpha}\right] L(\bar{x} - \bar{x}') = -s_b\delta(\bar{x} - \bar{x}').$$

Here, $s_b = 2K/(\kappa_0 S_0)^2 a^2 = (4\pi q^2 N/a^2\gamma_b^3)/(\kappa_0 S_0 v_b)^2$ is a measure of the beam space-charge intensity.

- For the beam confined by the external focusing force the maximum value of $(s_b)^{max} \sim 1$.

Perturbative Hamiltonian Transformation Method

- We search for a time-dependent canonical transformation of the form $(x^\alpha, p^\alpha, H, s) \rightarrow (Q^\alpha, P^\alpha, K, s)$

$$x^\alpha = x^\alpha(Q, P, s),$$
$$p^\alpha = p^\alpha(Q, P, s),$$

with time-independent transformed Hamiltonian $K(Q, P)$.

- For every canonical transformation there is a function S that satisfies the differential relation

$$[p^\alpha dx^\alpha] - H ds = dS + [P^\alpha dQ^\alpha] - K ds.$$

- Express $S = U + p_0(Q, P, s)^\alpha (x - Q)^\alpha$, where $U(Q, P, s)$ and $p_0(Q, P, s)$ are functions of the new phase-space variables.

$$\left[(x - Q)^\alpha \frac{\partial p_0^\alpha}{\partial P^\beta}\right] = \left[(p - p_0)^\alpha \frac{\partial (x - Q)^\alpha}{\partial P^\beta}\right] - \frac{\partial U}{\partial P^\beta},$$
$$(p - P)^\beta = -\left[(p - p_0)^\alpha \frac{\partial (x - Q)^\alpha}{\partial Q^\beta}\right] + \frac{\partial U}{\partial Q^\beta} + \left[(x - Q)^\alpha \frac{\partial p_0^\alpha}{\partial Q^\beta}\right],$$
$$K - H = -(p - p_0)^\alpha \frac{\partial (x - Q)^\alpha}{\partial s} + \frac{\partial U}{\partial s} + (x - Q)^\alpha \frac{\partial p_0^\alpha}{\partial s}.$$

The distribution function in the new coordinates

- Particle conservation in the phase-space volume $DxDp$ under the transformation

$$F(Q, P, s)\ DQDP = f(x, p, s)\ DxDp.$$

- For a canonical transformation, the phase-space volume is conserved according to $DxDp = DQDP$, and therefore $F(P, Q, s) = f[x(Q, P, s), p(Q, P, s), s]$.

- The new distribution function satisfies the Vlasov equation $dF/ds = 0$.

- For a time-independent Hamiltonian, there exists a trivial solution to the Vlasov equation, $F = G[K(Q, P)]$ for arbitrary function G.

- The matched solution can be found from $f(x, p, s) = G\{K_G[Q_G(x, p, s), P_G(x, p, s)]\}$.

- For solutions of this form, the Hamiltonian becomes

$$H(x, p, s) = \frac{\kappa(s)[\eta^\alpha x^\alpha x^\alpha]}{2} + \epsilon\{\frac{[p^\alpha p^\alpha]}{2} +$$
$$+ \int L[x, x(\bar{Q}, \bar{P}, s)]G[K(\bar{Q}, \bar{P})]D\bar{Q}D\bar{P}\}.$$

Iterative procedure of finding the canonical transformation in terms of the small parameter $\epsilon \sim \sigma_v/2\pi \ll 1$

- Make an expansion

$$p = p_0(Q,P,s) + \sum_{n=1} \epsilon^n p_n(Q,P,s),$$

$$x = Q + \sum_{n=1} \epsilon^n x_n(Q,P,s),$$

$$U = U_0(Q,P,s) + \sum_{n=1} \epsilon^n U_n(Q,P,s),$$

$$K = K_0(Q,P) + \sum_{n=1} \epsilon^n K_n(Q,P),$$

where p_n, x_n, U_n and K_n $(n = 0, 2, ...)$ are functions to be determined by the iterative procedure.

- We expand the function H according to

$$H(x,p,s) \equiv H[Q + \sum_{n=1} \epsilon^n x_n, p_0 + \sum_{n=1} \epsilon^n p_n, s]$$

$$= H_0(Q,P,s) + \sum_{n=1} \epsilon^n H_n(Q,P,s).$$

Iterative procedure up to second order in $\epsilon \sim \sigma_v/2\pi$

- For the Hamiltonian function $H(x, p, s)$ the functions Z_n depend only on p_k and x_k, with $k < n$.

- Because $K_0 = 0$, the average Hamiltonian K has the form $K = \epsilon(K_1 + \epsilon K_2 + \epsilon^2 K_3 + \cdots)$. The ϵ in front of the bracket renormalizes the time scale, so that the average dynamics occurs on the slow time-scale $Q = Q(\epsilon s)$ and $P = P(\epsilon s)$.

- Therefore, to determine the trajectories $x(s)$ and $p(s)$ valid to second order in ϵ, we need to determine the average Hamiltonian K valid up to the third order in ϵ.

Canonical transformation up to second order in $\epsilon \sim \sigma_v/2\pi$

$$x^\alpha = Q^\alpha - \epsilon\kappa^{(2)}\eta^\alpha Q^\alpha + \epsilon^2\left\{2\kappa^{(3)}\eta^\alpha P^\alpha + (\kappa\kappa^{(2)})^{(2)}Q^\alpha\right\},$$

and

$$p^\alpha = \left\{P^\alpha - \kappa^{(1)}\eta^\alpha Q^\alpha\right\} + \epsilon\left\{\kappa^{(2)}\eta^\alpha P^\alpha + (\kappa\kappa^{(2)})^{(1)}Q^\alpha\right\} +$$
$$+\epsilon^2\Big\{\Big(3 < (\kappa^{(2)})^2 > -2(\kappa\kappa^{(3)})^{(1)} - (\kappa\kappa^{(2)})^{(2)}\Big)P^\alpha$$
$$+\Big(\kappa^{(3)} < (\kappa^{(1)})^2 > -(\kappa(\kappa\kappa^{(2)})^{(2)})^{(1)}\Big)\eta^\alpha Q^\alpha\Big\} + \epsilon\frac{\partial}{\partial Q^\alpha}\Psi(Q)^{(1)}.$$

- Here, $< A >\equiv (1/S)\int_s^{s_0+S} d\bar{s}A(\bar{s})$, and $\ll A \gg\equiv A- < A >$. Here, we also introduce the notation $A^{(0)} \equiv\ll A \gg$ and

$$A^{(n)} \equiv\ll \int dsA^{(n-1)} \gg, \quad \text{for } n \geq 1$$

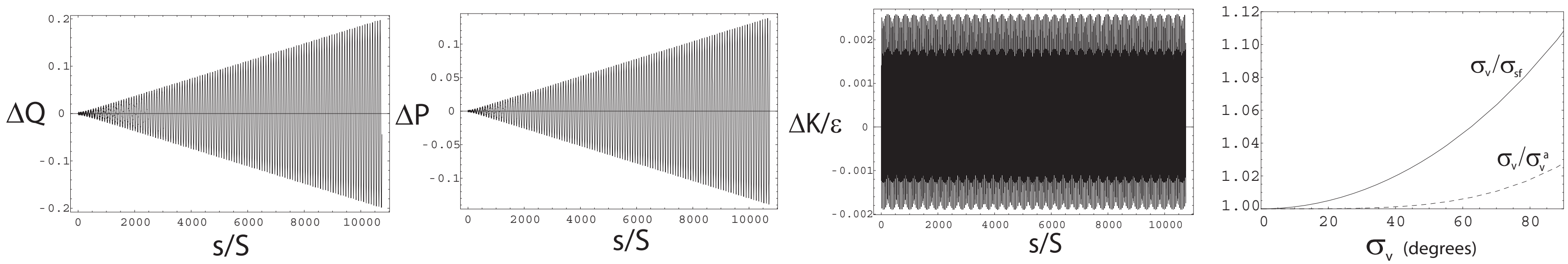
- The time-independent Hamiltonian correct to the third order in ϵ is

$$K = \epsilon\Big\{\frac{[P^\alpha P^\alpha]}{2}\Big(1 + 3\epsilon^2 < (\kappa^{(2)})^2 >\Big) + \frac{[Q^\alpha Q^\alpha]}{2}\Big(< (\kappa^{(1)})^2 >$$
$$+\epsilon^2 < ((\kappa\kappa^{(2)})^{(1)})^2 > \Big)+ < \Psi(Q) > \Big\}$$

$$\Psi(Q) = \int D\bar{Q}L(Q_\alpha(1 + \epsilon\eta^\alpha\kappa^{(2)}), \bar{Q}_\alpha(1 + \epsilon\eta^\alpha\kappa^{(2)}))n(\bar{Q})$$

Approximate vs. Exact

- Plots of $\Delta Q(s) = Q(s) - Q_{tr}(x,p,s)$, $\Delta P(s) = P(s) - P_{tr}(x,p,s)$ and $\Delta K/\epsilon = [K - K_{tr}(x,p,s)]/\epsilon$ as functions of the normalized variable s/S over the interval $[0, 1/\epsilon^4]$ for the choice of dimensionless parameters $s_b = 0$, $\bar{\kappa} = k = 1$, $\epsilon = 0.1$ ($\sigma_v = 25^0$).



- The approximate expression for the square of the vacuum phase advance σ_v^2 valid up to forth order in small parameter ϵ, i.e.,

$$\sigma_v^2 = S^2\epsilon^2 \left\{1 + 3\epsilon^2 < (\kappa^{(2)})^2 >\right\} \left\{ < (\kappa^{(1)})^2 > + \epsilon < \kappa(\kappa^{(2)})^2 > + \epsilon^2 < ((\kappa\kappa^{(2)})^{(1)})^2 >\right.$$

- Plots of the normalized quantities σ_v/σ_{sf} and σ_v/σ_v^a versus the vacuum phase advance σ_v for periodic step-function lattice with filling factor $\delta = 1/2$.

Evaluation of self-field potential

- Expanding correct to second order in the small parameter ϵ, we obtain the expression for the average potential

$$
\begin{aligned}
< \Psi(Q) > \quad &= (1+\epsilon^2 < v^2 >)\phi_0 + \epsilon^2 < v^2 > \Big(\phi_1 - \left[\eta^\alpha Q^\alpha \frac{\partial}{\partial Q^\alpha}\right]\phi_2 \\
&+ \frac{1}{2}\left[\eta^\alpha \eta^\beta Q^\alpha Q^\beta \frac{\partial^2}{\partial Q^\alpha \partial Q^\beta}\right]\phi_0\Big),
\end{aligned}
$$

where the functions $\phi_0(Q)$, $\phi_1(Q)$ and $\phi_2(Q)$ satisfy the Poisson-type equations

$$
\begin{aligned}
\nabla_\perp^2 \phi_0 &= -s_b n(Q), \\
\nabla_\perp^2 \phi_1 &= -s_b \left(\left[Q^\alpha \frac{\partial}{\partial Q^\alpha}\right] + \frac{1}{2}\left[\eta^\alpha \eta^\beta Q^\alpha Q^\beta \frac{\partial^2}{\partial Q^\alpha \partial Q^\beta}\right]\right) n(Q), \\
\nabla_\perp^2 \phi_2 &= -s_b \left[\eta^\alpha Q^\alpha \frac{\partial}{\partial Q^\alpha}\right] n(Q),
\end{aligned}
$$

- In cylindrical coordinates $Q_1 = r\cos(\theta), Q_2 = r\sin(\theta)$

$$
\left[\eta^\alpha Q^\alpha \frac{\partial}{\partial Q^\alpha}\right] = Q^1 \frac{\partial}{\partial Q^1} - Q^2 \frac{\partial}{\partial Q^2} = \cos(2\theta) r \frac{\partial}{\partial r} - \sin(2\theta)\frac{\partial}{\partial \theta}.
$$

Boundary conditions for self-field potential

- Needs to specify some boundary surface in the coordinate space (Q^1, Q^2) and certain boundary conditions on this boundary.

- It is convenient to designate this boundary surface to be a surface in the coordinate space (Q^1, Q^2), where the function $L(Q, \bar{Q})$ satisfies the same boundary conditions as the function $L(x, \bar{x})$ in the coordinate space (x^1, x^2).

- In that case, the boundary conditions for $\phi_0(Q)$, $\phi_1(Q)$ and $\phi_2(Q)$ in the coordinate space (Q^1, Q^2) are the same as the boundary conditions for the Green's function $L(Q, \bar{Q})$.

- Note that this boundary surface in the coordinate space (Q^1, Q^2) becomes a surface that oscillates around the boundary surface in the coordinate space (x^1, x^2).

- Because the two surfaces differ, the average potential $\bar{\Psi}(Q)$ in the coordinate space (Q^1, Q^2) does not satisfy the same boundary conditions as the un-averaged potential in the coordinate space (x^1, x^2).

Equilibrium self-field potential

- In equilibrium

$$n(Q) = \int dPG(K) = \int dPG(K_{kin} + K_{ext} + \Phi_0 + \epsilon^2 < v^2 > \bar{\Psi}_1).$$

- For perfectly conducting cylindrical boundary of radius R_w,

$$\bar{\Psi}_1 = p(r) + \cos(4\theta) q(r),$$

where

$$\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} + s_b n_0'(r)\right) p(r) = -s_b \frac{8}{R_w^4} \int_0^{R_w} dr r^3 n_0(r),$$

$$\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - \frac{16}{r^2} + s_b n_0'(r)\right) q(r) = -2s_b \left(n_0 + \frac{4}{r^2}\int_0^r d\bar{r}\bar{r} n_0(\bar{r}) - \frac{12}{r^4}\int_0^r d\bar{r}\bar{r}^3 n_0(\bar{r})\right)$$

- with boundary conditions

$$p(R_w) = -\frac{2s_b}{R_w^2}\int_0^{R_w} dr r^3 n_0(r), \quad q(R_w) = -s_b\left(\frac{2}{R_w^2}\int_0^{R_w} d\bar{r}\bar{r}^3 n_0(\bar{r}) - \frac{1}{2}\int_0^{R_w} d\bar{r}\bar{r} n_0(\bar{r})\right).$$

- Here, $n_0(r) \equiv \int dPG[K_{kin} + K_{ext} + \Phi_0(r)]$ and $n_0' \equiv \partial n_0(r)/\partial\Phi_0$ with $\Phi_0(r)$ determined self-consistently through

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\Phi_0 = -s_b n_0(r),$$

Thermal equilibrium distribution

- Zero-order thermal equilibrium average beam profile

$$\bar{n}_0 = n_0(r)/n_0(0) = \exp\left(-\frac{k^2 r^2/2 + \Phi_0(r)}{\bar{T}}\right).$$

- For thermal equilibrium distribution normalized quantities are function of two parameters

$$\epsilon \sim (\sigma_v/2\pi)$$

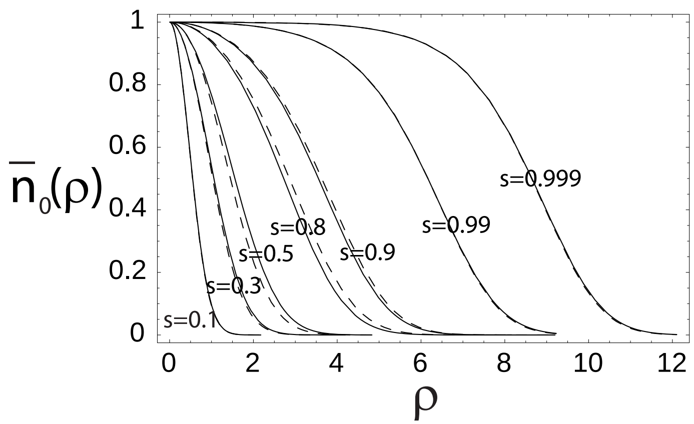
and beam intensity

$$\bar{s} = \omega_p^2/2\omega_{sf}^2 < 1,$$

where ω_p is the plasma density at the beam center and $\omega_{sf} = (\sigma_v/2\pi)(v_b/S)$ is the average focusing frequency of the lattice.

Maxwellian distribution averaged density profile

- Plots of density profiles $\bar{n}_0(\rho)$ for different values of intensity parameter $\bar{s} = 0.1; 0.3; 0.5; 0.8; 0.9; 0.99; 0.999$.



- For $\bar{s} \ll 1$

$$\bar{n}_0(\rho) \approx \left(1 + \frac{\rho^2}{4\bar{s}}\right)^{\bar{s}} \exp\left(-\frac{\rho^2}{4\bar{s}}\right).$$

- For $\Delta = 1/\bar{s} - 1 \ll 1$

$$\bar{n}_0(\rho) \approx \frac{\left[1 + \frac{1}{2}\Delta + \frac{1}{24}\Delta^2\right]^2}{\left[1 + \frac{1}{2}\Delta I_0(\rho) + \frac{1}{24}[\Delta I_0(\rho)]^2\right]^2},$$

where $I_0(\rho)$ is zero order modified Bessel function.

Corrections to density profile for Maxwellian distribution

- Define RMS radius as

$$R^2(\theta) = \frac{\int \rho^3 \bar{n}(\rho,\theta) d\rho}{\int \rho \bar{n}(\rho,\theta) d\rho}, \qquad R_0^2 = \frac{\int \rho^3 \bar{n}_0(\rho) d\rho}{\int \rho \bar{n}_0(\rho) d\rho}.$$

- Relative change in RMS radius as $\delta R(\theta)/R_0 = R(\theta)/R_0 - 1$ can be expressed as

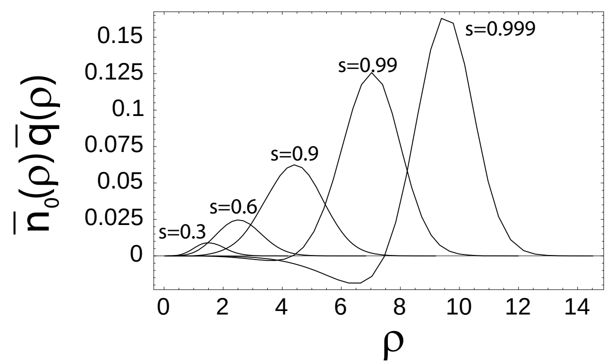
$$\frac{\delta R(\theta)}{R_0} = -\epsilon^2 < v^2 > \cos(4\theta) A(\bar{s}),$$

where $A(\bar{s})$ is given by

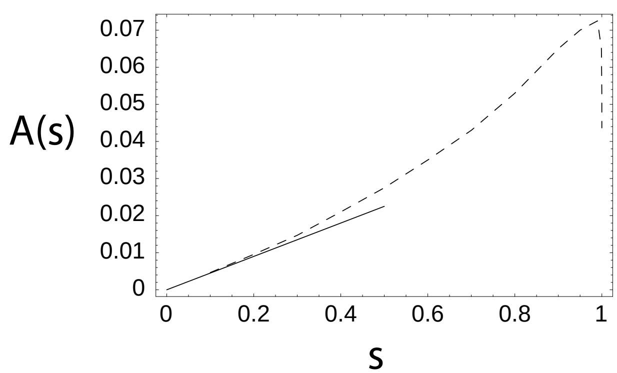
$$A(\bar{s}) = \frac{1}{2}\left[\frac{\int_0^\infty \rho^3 \bar{q}(\rho)\bar{n}_0(\rho) d\rho}{\int_0^\infty \rho^3 \bar{n}_0(\rho) d\rho} - \frac{\int_0^\infty \rho \bar{q}(\rho)\bar{n}_0(\rho) d\rho}{\int_0^\infty \rho \bar{n}_0(\rho) d\rho}\right].$$

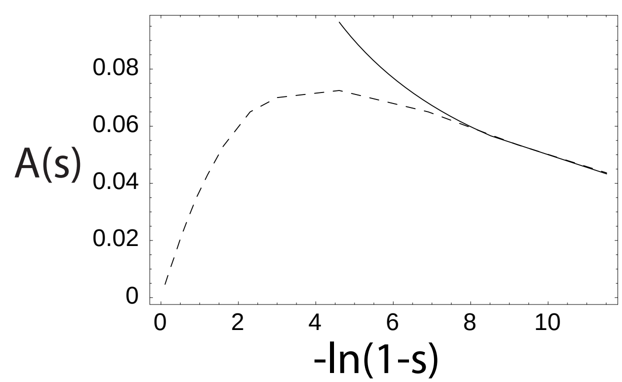
Corrections to density profile, cont'd

- Plots of density profile correction $\bar{q}(\rho)\bar{n}_0(\rho)$ for different values of intensity parameter $\bar{s} = 0.3; 0.6; 0.9; 0.99; 0.999$.



- Plots of function $A(\bar{s})$, (a) linear s-scale and (b) logarithmic s-scale.





- Here $\rho_{1/2}(\bar{s}) \simeq \ln\left[\frac{C}{\Delta}\sqrt{2\pi\frac{C}{\Delta}}\right]$ is the beam edge radius $\bar{n}_0(\rho_{1/2}) = 1/2$ and $C \approx 0.78$.

Specific example of the beam inside of perfectly conducting cylindrical boundary

- Kapchinskij-Vladimirskij distribution $G(K) = (\bar{n}_0/2\pi)\delta(K - K_0)$ with

$$n_0(r) = \begin{cases} \bar{n}_0, & r < a, \\ 0, & a < r \leq R_w. \end{cases}$$

- Beam radius r_b becomes weakly dependent on the angle θ according to

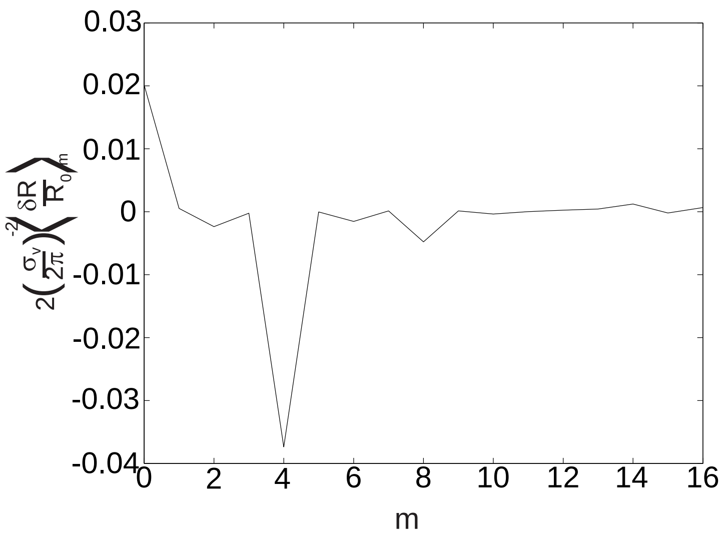
$$r_b(\theta) = a\left\{1 + \epsilon^2 < v^2 > \frac{\bar{s}}{(1-\bar{s})}\left(\frac{(a/R_w)^4}{1 - \frac{2\bar{s}}{1-\bar{s}}\ln\frac{a}{R_w}} + \frac{\cos(4\theta)}{2}\frac{(a/R_w)^8}{1 + \frac{\bar{s}}{4(1-\bar{s})}\left(1 - \left(\frac{a}{R_w}\right)^8\right)}\right)\right\}$$

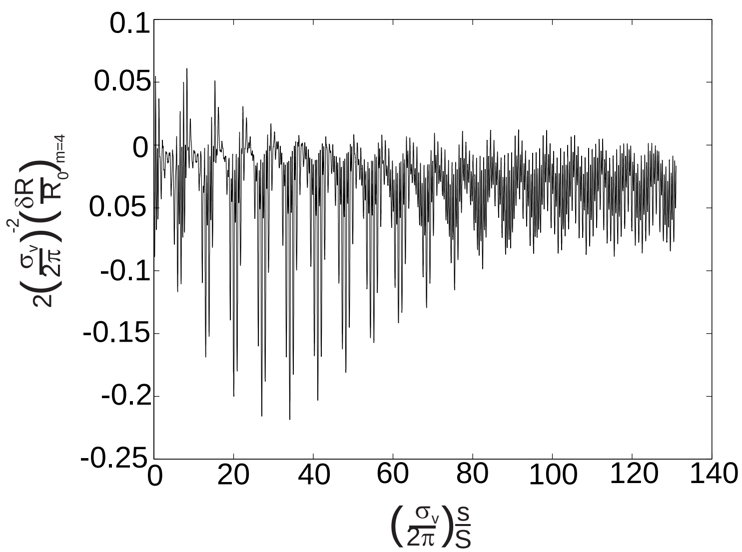
- For $R_w \to \infty$, $\bar{\Psi}_1 = 0$, and the total self-field potential inside the beam is given by zero order potential Φ_0, which is what one would expect for a Kapchinskij-Vladimirskij distribution in free space ($R_w \to \infty$) which generates the constant beam density and linear self-field forces.

- For finite R_w, the image charge oscillations produce additional contributions to the average self-field potential inside the beam, which lead to the octopole correction to the average beam radius.

Simulation studies

- The simulations results for a sinusoidal lattice $\kappa(s) = \kappa_0 \sin(2\pi s/S)$
- Beam intensity $\bar{s}_b = 0.5$, wall radius $R_w = 5R_0$, and $\sigma_v = 25^0$ which corresponds to $\epsilon = 0.07$.
- Fourier spectrum $\sim exp(im\theta)$ of the averaged over time relative RMS radius change $< \delta R(\theta)/R_0 >$ as a function of azimuthal mode number m.





- Time history of $m = 4$ component of $\delta R(\theta, s)/R_0$.

Results and Conclusions

- New Hamiltonian averaging technique has been developed, which incorporated both the applied periodic focusing force and the self-field force of the beam particles.

- Using the particle's vacuum phase advance $\epsilon = \sigma_v/2\pi$ treated as a small parameter, the perturbative series are constructed to transform away the fast particle orbit oscillations and obtain the average Hamiltonian accurate to order ϵ^3.

- The higher order corrections allow us to extend the average formulaic results to larger vacuum phase advances approaching $\sigma_v \sim 90^0$ with accuracies to within several percent.

- The equations determining the average self-field potential have been derived for general boundary conditions by taking into account the average contribution of the charges induced on the boundary.

- The average equation can be used to find an approximate invariant of the original system, and can be used to determine self-consistent beam quasi-equilibrium solutions that are matched to the focusing channel.

- The time-dependent formulation can be used to describe collective beam dynamics which is slow in the transformed coordinates (slow compared to the period of of the lattice).

Results and Conclusions, cont'd

- The corrections to the self-field are small, and therefore the "smooth-focusing" approximation for the self-field potential can be a good approximation even for moderate values of the vacuum phase advance.

- For example, for vacuum phase advance of $\sigma_v = 90^0$ the correction to the RMS radius of the beam described by a thermal equilibrium distribution arising from the corrections to the average self-field potential is of order 0.5%.

- Nonetheless, note that because the average self-field potential acquires an octupole component, the average motion of some beam particles becomes non-integrable and the trajectories become chaotic.

- This chaotic behavior of some of the beam particles may change the nature of the Landau damping (or growth) of collective excitations supported by the beam.

- Also, due to the presence of the extra non-axisymmetric terms in the equations for the self-field potential, the stability properties of different beam quasi-equilibria can change significantly.