THEORETICAL STUDY OF TRANSVERSE-LONGITUDINAL EMITTANCE COUPLING*

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Abstract

The effect of a weakly coupled periodic lattice in terms of achieving emittance exchange between the transverse and longitudinal directions is investigated using the generalized Courant-Snyder theory for coupled lattices.

INTRODUCTION

Recently, the concept and technique of transverse-longitudinal emittance coupling have been proposed for applications in the Linac Coherent Light Source [1, 2] and other free-electron lasers to reduce the transverse emittance of the electron beam. Such techniques can also be applied to the driver beams for the heavy ion fusion and beam-driven high energy density physics, where the transverse emittance budget is typically tighter than the longitudinal emittance. The proposed methods consist of one or several coupling components which completely swap the emittances of one of the transverse directions and the longitudinal direction at the exit of the coupling components. The complete emittance exchange is realized in one pass through the coupling components. In the present study, we investigate the effect of a weakly coupled periodic lattice in terms of achieving emittance exchange between the transverse and longitudinal directions. A weak coupling component is introduced at every focusing lattice, and we would like to determine if such a lattice can realize the function of emittance exchange.

For simplicity, we will only study the coupling between one of the transverse directions, the $x-$direction, and the longitudinal direction, the $z-$direction. The focusing lattice in the $x-$direction is a periodic FODO lattice specified by a focusing coefficient $\kappa_q(s)$, where s is the distance along the longitudinal direction. The longitudinal dynamics is modelled by a synchrotron oscillation with a constant synchrotron focusing coefficient κ_z. In every FODO lattice, two small-amplitude transverse-longitudinal coupling components are introduced. As discussed in Refs. [1, 2], such components can be realized by a dipole mode cavity, which generates a longitudinal acceleration force proportional to the transverse displacement, and a transverse acceleration force proportional to the longitudinal position relative to the beam centroid. The coupling can be viewed as a skew-quad between the $x-$direction and $z-$direction. The coupling focusing strength will be represented by $\kappa_s(s)$.

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We will study the emittance dynamics from the viewpoint of the beam covariance matrix

$$\sigma \equiv \begin{pmatrix} \langle x^2\rangle & \langle xz\rangle & \langle xp_x\rangle & \langle xp_z\rangle \\ \langle xz\rangle & \langle z^2\rangle & \langle zp_x\rangle & \langle zp_z\rangle \\ \langle p_x x\rangle & \langle p_x z\rangle & \langle p_x^2\rangle & \langle p_x p_z\rangle \\ \langle p_z x\rangle & \langle p_z z\rangle & \langle p_x p_z\rangle & \langle p_z^2\rangle \end{pmatrix},$$

where $\langle\rangle \equiv \int f_b dxdzdp_xdp_z$ represents average over the particle distribution function. After the beam propagates through the coupled lattice, the covariance matrix is transformed to

$$\sigma(s) = M(s)^\dagger \sigma_0 M(s),$$

where $M(s)$ is the transfer matrix from $s = 0$ to $s = s$, $\sigma_0 = \sigma(0)$, and $M(s)^\dagger$ is the transpose of $M(s)$. The emittance dynamics in the coupled lattice is therefore completely specified by the transfer matrix $M(s)$.

GENERALIZED COURANT-SNYDER THEORY AND REPRESENTATION FOR COUPLED DYNAMICS

We will use the recently developed generalized Courant-Snyder (CS) theory and parameterization method for coupled lattices to parameterize the transfer matrix [3, 4, 5, 6, 7]. The main result of the theory is summarized as follows. The Hamiltonian for the coupled transverse dynamics is given by

$$H = u^\dagger A u\,,\ u = (x, z, p_x, p_z)^\dagger\,, \tag{1}$$

$$A = \begin{pmatrix} \kappa & 0 \\ 0^\dagger & \dfrac{I}{2} \end{pmatrix},\ \kappa = \begin{pmatrix} \kappa_q/2 & \kappa_s \\ \kappa_s & \kappa_z/2 \end{pmatrix}. \tag{2}$$

Here, the 2×2 matrix $\kappa(t)$ is time-dependent and symmetric, and I is the 2×2 unit matrix. The transverse and longitudinal dynamics are coupled through the $\kappa_s(t)$ term. The solution of the linear coupled system corresponding to H is given by a transfer matrix $M(t)$, which is a time-dependent 4×4 symplectic matrix [8]. Because there are 10 free parameters for a 4×4 symplectic matrix, many different mathematical parameterization schemes for $M(t)$ exist. Teng and Edwards [9, 10, 11] first systematically studied the transfer matrix and derived various parameterization schemes [9], among which the "symplectic rotation form" [10] has been adopted in lattice design and particle tracking codes. Other possible parameterizations have also been considered [12, 13]. The generalized Courant-Snyder theory adopted here gives a complete description of the coupled transverse dynamics, and has the same structure as the original CS theory for one degree of freedom. The four basic components of the original CS theory that have physical importance, *i.e.*, the envelope equation, phase advance, transfer matrix, and the CS invariant, all have their counterparts, with remarkably similar expressions, in the generalized CS theory. The unique feature of the generalized CS theory is the non-Abelian (non-commutative) nature of the theory. In the generalized theory, the envelope function w is generalized to an envelope matrix, and the envelope equation becomes a matrix envelope equation with matrix operations that are not commutative. The generalized Courant-Snyder theory gives a parameterization of the 4D symplectic transfer matrix M [Eqs. (3)] that has the same structure as the parameterization of the 2D symplectic transfer matrix [Eq. (7)] in the original CS theory. The transfer matrix M is given by

$$M\left(s\right) = SPS_0^{-1},\ P = \begin{pmatrix} P_1 & -P_2 \\ P_2 & P_1 \end{pmatrix}, \tag{3}$$

$$S = \begin{pmatrix} w^{\dagger} & 0 \\ w^{-1}\dot{w}w^{\dagger} & w^{-1} \end{pmatrix},\ S_0^{-1} = \begin{pmatrix} w_0^{-1\dagger} & 0 \\ -\dot{w}_0 & w_0 \end{pmatrix}. \tag{4}$$

In Eqs. (3) and (4), both S and P are symplectic matrices, and w is the 2×2 envelope matrix satisfying the envelope matrix equation

$$\ddot{w} + w\kappa = \left(w^{-1}\right)^{\dagger} w^{-1} \left(w^{-1}\right)^{\dagger}. \tag{5}$$

The P matrix is determined from the differential equation

$$\dot{P} = P\dot{\phi},\ \dot{\phi} \equiv \begin{pmatrix} +0 & -\left(w^{-1}\right)^{\dagger} w^{-1} \\ \left(w^{-1}\right)^{\dagger} w^{-1} & 0 \end{pmatrix}, \tag{6}$$

which admits solutions of the form $P = \begin{pmatrix} P_1 & P_2 \\ -P_2 & P_1 \end{pmatrix}$.

For comparison, we recall that for the uncoupled (one degree of freedom) dynamics, the transfer matrix $M\left(t\right)$ is a 2×2 symplectic matrix with following decomposition

$$M = \begin{pmatrix} w & 0 \\ \dot{w} & \frac{1}{w} \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} w_0^{-1} & 0 \\ -\dot{w}_0 & w_0 \end{pmatrix}. \tag{7}$$

The envelope function $w\left(t\right)$ here is a scalar and satisfies the nonlinear envelope equation $\ddot{w} + \kappa_q\left(t\right) w = w^{-3}$.

For the coupled case, we can readily show that $PP^{\dagger} = I$, and $Det\left(P\right) = 1$ from the fact that P belongs to $Sp\left(4, R\right)$. Therefore, P corresponds to a rotation in the 4D phase space, $P \in SO\left(4\right)$. In this sense, $P^{\dagger}$ is the 4D non-Abelian generalization of the 2D rotation matrix in the expression for the transfer matrix M for the original Courant-Snyder theory, *i.e.*, the second term on the right-hand side of Eq. (7). Because $\dot{\phi}^{\dagger} = -\dot{\phi}$, it follows that $\dot{\phi}$ belongs to the Lie algebra $so\left(4\right)$, *i.e.*, $\dot{\phi}$ is an infinitesimal generator of a 4D rotation. In other words, $\dot{\phi}$ is an "angular velocity" in 4D space, which is equivalent to a phase advance rate in 4D space. The 4D phase advance rate $\dot{\phi}$ is determined from

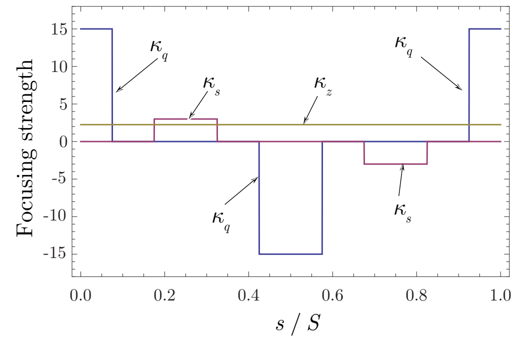

Figure 1: Focusing strength.

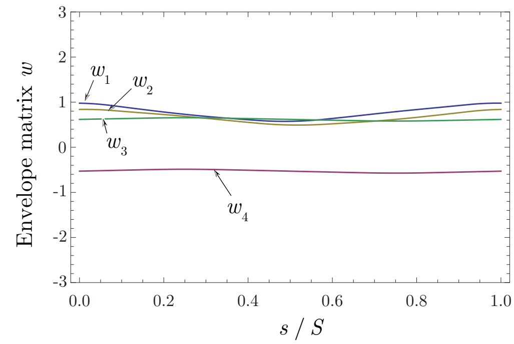

Figure 2: Matched solution of the envelope matrix.

the 2×2 matrix $\beta^{\dagger -1} = \left(w^{-1}\right)^{\dagger} w^{-1}$, which is remarkably similar to the phase advance rate $\beta^{-1} = 1/w^2$ in the original Courant-Snyder theory for one degree of freedom. For both cases, the transfer maps consist of a scaling matrix and the a rotation matrix.

As in the case of uncoupled dynamics, the transfer matrix for the coupled case does not match the lattice period. However, the transfer matrix is regular. It does not match the lattice period simply because the phase advance is not 2π in one lattice period. There is no need to numerically solve for the transfer matrix for many periods along the beam path. We only need to numerically find a matched solution of the envelope matrix in one lattice period and then carry out the calculation of the phase advance along the beam path from Eq. (6) to find the transfer matrix. This is one of the practical values of the generalized CS parameterization for coupled dynamics. The definition of Twiss parameters for the coupled dynamics have the same structure as the uncoupled case, $\beta = w^{\dagger} w$, $\alpha = -w^{\dagger} w'$, and $\gamma = \left(w^{\dagger} w\right)^{-1} + w'^{\dagger} w'$. All of the Twiss parameters are periodic functions of the coupled lattice. They are fixed once the coupled lattice is set up.

EMITTANCE DYNAMICS

We now apply the generalized CS parameterization method to the coupled lattice between the $x-$direction and the $z-$direction, and investigate how the emittance and covariance matrix evolves. One of the two indepen-

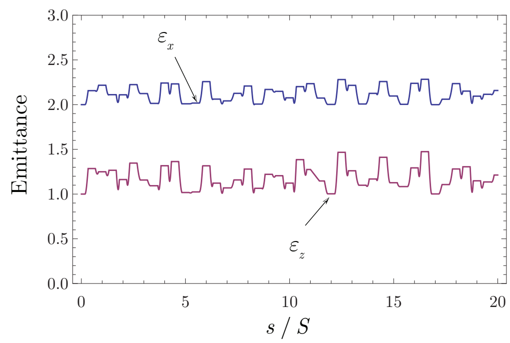

Figure 3: Emittance dynamics of ϵ_x and ϵ_z.

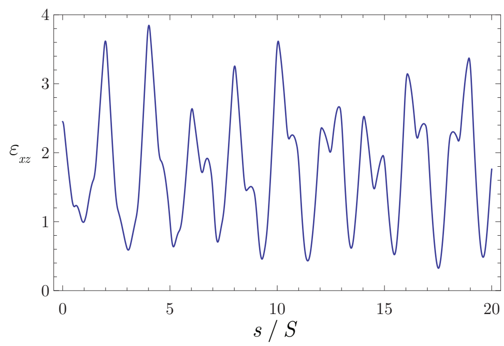

Figure 4: Emittance dynamics of ϵ_{xz}.

dent [14] 4D emittances is typically defined as the square root of the determinant of σ, which is a constant, *i.e.*, $\epsilon_{4D} = \sqrt{Det[\sigma]} = \epsilon_{4D}(s = 0)$, because $Det[M] = 1$. However, the dynamics of the determinants of submatrices of σ can be complicated. The often studied $x-$emittance and $z-$emittance squared are defined as the determinants of the corresponding submatrices

$$\epsilon_x^2 \equiv Det\begin{pmatrix} \langle x^2 \rangle & \langle xp_x \rangle \\ \langle xp_x \rangle & \langle p_x^2 \rangle \end{pmatrix}, \qquad (8)$$

$$\epsilon_z^2 \equiv Det\begin{pmatrix} \langle z^2 \rangle & \langle zp_z \rangle \\ \langle zp_z \rangle & \langle p_z^2 \rangle \end{pmatrix}. \qquad (9)$$

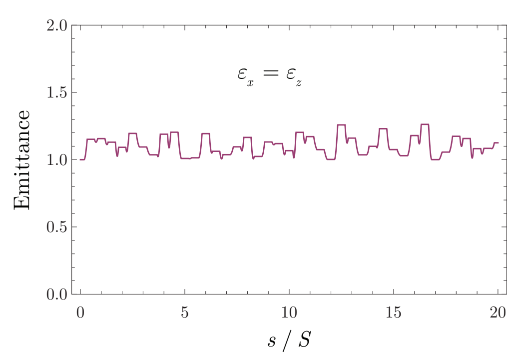

Figure 5: Emittance dynamics for $\epsilon_{x0} = \epsilon_{z0}$.

As an example, we investigate the coupled lattice plotted in Fig. 1. The matched solution of the envelope matrix w is shown in Fig. 2. The dynamics of ϵ_x and ϵ_z are displayed in Fig. 3 for the case where $\epsilon_{x0} = 2$ and $\epsilon_{z0} = 1$ (in arbitrary units). As expected, the dynamics of ϵ_x and ϵ_z do not match the lattice period, and both ϵ_x and ϵ_z vary with s. For this case, the coupling does not induce an exchange between ϵ_x and ϵ_z. However, this effect is partially attributed to the fact that the definitions of ϵ_x and ϵ_z in Eqs. (8) and (9) do not contain all of the relevant information about the emittance for the coupled dynamics. For instance, we should also examine the following determinant

$$\epsilon_{xz}^2 \equiv Det(\sigma_{xz}), \; \sigma_{xz} \equiv \begin{pmatrix} \langle x^2 \rangle & \langle xz \rangle \\ \langle xz \rangle & \langle z^2 \rangle \end{pmatrix}, \qquad (10)$$

which measures the area of the covariance ellipse of the beam determined by eigenvalues and eigenvectors of σ_{xz}. Because the off-diagonal terms in σ_{xz} are non-vanishing, the ellipse is tilted. This is of course induced by the coupling. In Fig. 4 the dynamics of σ_{xz} is plotted, which indicates that σ_{xz} oscillates with time. But on average, ϵ_{xz} decreases compared with the uncoupled case.

It can be shown [2] that if $\epsilon_{x0} = \epsilon_{z0}$ initially, then the coupling does not induce any exchange between ϵ_x and ϵ_z, and $\epsilon_x = \epsilon_z$ for all time. Nevertheless, this does not mean that $\epsilon_x = \epsilon_{x0}$ and $\epsilon_z = \epsilon_{z0}$. In Fig. 5, such a case is displayed. Even though $\epsilon_x = \epsilon_z$ for all the time, ϵ_x and ϵ_z are not constant. They both increase.

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