AMPLIFICATION OF CURRENT DENSITY MODULATION IN A FEL WITH AN INFINITE ELECTRON BEAM*

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Abstract

We show that the paraxial field equation for a free electron laser (FEL) in an infinitely wide electron beam with $\kappa-2$ energy distribution can be reduced to a fourth ordinary differential equation (ODE). Its solution for arbitrary initial phase space density modulation has been derived in the wave-vector domain. For initial current modulation with Gaussian profile, close form solutions are obtained in space-time domain.

INTRODUCTION

In developing an analytical model for a FEL-based coherent electron cooling system, an infinite electron beam has been assumed for the modulation and correction processes [1-3]. While the assumption has its limitation, it allows for an analytical close form solution to be obtained, which is essential for investigating the underlying scaling law, benchmarking the simulation codes and understanding the fundamental physics.

1D theory was previously applied to model a CeC FEL amplifier[4]. However, the theory ignores diffraction effects and does not provide the transverse profile of the amplified electron density modulation. On the other hand, 3D theories developed for a finite electron beam usually have solutions expanded over infinite number of modes determined by the specific transverse boundary conditions. Unless the mode with the largest growth rate substantially dominates other modes, both evaluation and extracting scaling laws can be complicated. Furthermore, it is also preferable to have an analytical FEL model with assumptions consistent with the other two sections of a CeC system.

Recently, we developed the FEL theory in an infinitely wide electron beam with $\kappa-1$ (Lorentzian) energy distribution[5]. Close form solutions have been obtained for the amplified current modulation initiated by an external electric field with various spatial-profiles. In this work, we extend the theory into $\kappa-2$ energy distribution and study the evolution of current density induced by an initial density modulation.

EQUATION OF MOTION

Assuming that the amplitude of the radiation field varies slowly with respect to the undulator period and that fast oscillation terms can be dropped, the amplitude of the radiation field is determined by the following paraxial field equation [6]

$$\begin{aligned}&\left(\nabla_\perp^2+2i\frac{\omega}{c}\frac{\partial}{\partial z}\right)\tilde{E}(\vec{r}_\perp,z)\\&=ij_0(\vec{r}_\perp)\int_0^z dz'\left[\frac{e\omega\theta_s^2}{2c^2\varepsilon_0}\tilde{E}(\vec{r}_\perp,z')+\frac{e}{\varepsilon_0\omega}\left(\nabla_\perp^2+2i\frac{\omega}{c}\frac{\partial}{\partial z}\right)\tilde{E}(\vec{r}_\perp,z')\right]\\&\times\int_{-\infty}^{\infty}e^{i\left(C+\frac{\omega}{c\gamma_z^2\mathrm{E}_0}P\right)(z'-z)}\frac{\partial}{\partial P}F(P)dP+\frac{i\theta_s\omega}{c\varepsilon_0}e\int_{-\infty}^{\infty}\tilde{f}_1(\vec{r}_\perp,P,0)e^{-i\left(C+\frac{\omega}{c\gamma_z^2\mathrm{E}_0}P\right)z}dP\end{aligned} \quad (1)$$

where $\tilde{E}(\vec{r}_\perp,z)$ is the complex amplitude of the radiation field, ω is the radiation frequency, C is the detuning, E_0 is the nominal electron energy, P is the electron energy deviation, θ_s is the electron deflection angle, $F(P)$ is the energy distribution function, $j_0(\vec{r}_\perp)$ is the transverse spatial distribution of the unperturbed electron beam and $\tilde{f}_1(\vec{r}_\perp,P,0)$ is the initial phase space density perturbation. Assuming $j_0(\vec{r}_\perp)=j_0$ is independent of $\vec{r}_\perp$ and performing Fourier transformation to eq. (1) lead to [5]

$$\begin{aligned}&\frac{\partial}{\partial\hat{z}}\tilde{R}(\hat{z},\hat{k}_\perp,\hat{C})\\&=\int_0^{\hat{z}}d\hat{z}'e^{i\hat{k}_\perp^2(\hat{z}-\hat{z}')}\left[\tilde{R}(\hat{z}',\hat{k}_\perp,\hat{C})+i\hat{\Lambda}_p^2\frac{\partial}{\partial\hat{z}'}\tilde{R}(\hat{z}',\hat{k}_\perp,\hat{C})\right]\int_{-\infty}^{\infty}d\hat{P}\frac{d\hat{F}}{d\hat{P}}e^{i(\hat{C}+\hat{P})(\hat{z}'-\hat{z})}\\&+\frac{e\theta_s}{2\varepsilon_0\Gamma}e^{i\hat{k}_\perp^2\hat{z}}\int_{-\infty}^{\infty}\tilde{f}_1(\vec{k}_\perp,\hat{P},0)e^{-i(\hat{C}+\hat{P})\hat{z}}d\hat{P}\end{aligned} \quad (2)$$

where

$$\tilde{R}(z,k_\perp,C)\equiv e^{i\frac{k_\perp^2 c}{2\omega}z}\tilde{E}(z,k_\perp,C). \quad (3)$$

In eq. (2), we used the normalized variables as defined in [6] and [5], i.e. $\hat{z}=\Gamma z$, $\hat{C}=C/\Gamma$, the 1D- gain parameter

$$\Gamma\equiv\left[\frac{\pi j_0\theta_s^2\omega}{c\gamma_z^2\gamma I_A}\right]^{1/3}, \quad (4)$$

the Alfven current $I_A\equiv m_e c^3/e$, the pierce parameter $\rho=\gamma_z^2\Gamma c/\omega$, the space charge parameter

$$\hat{\Lambda}_p\equiv\frac{1}{\Gamma^2}\left[\frac{4\pi j_0}{\gamma_z^2\gamma I_A}\right]^{1/2}, \quad (5)$$

the normalized transverse wave vector

$$\vec{\hat{k}}_\perp\equiv\sqrt{\frac{\rho}{2}}\frac{\vec{k}_\perp}{\gamma_z\Gamma}, \quad (6)$$

and the normalized energy distribution function $\hat{F}(\hat{P})$ satisfying

$$\int_{-\infty}^{\infty}\hat{F}(\hat{P})d\hat{P}=1.$$

We assume the following $\kappa-2$ energy distribution,

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$$F(\hat P)=\frac{2}{\pi\hat q}\frac{1}{\left(1+\frac{\hat P^2}{\hat q^2}\right)^2}=-\hat q^2\frac{\partial}{\partial\hat q}\left(\frac{1}{\hat q}F_1(\hat P)\right). \quad (7)$$

Applying the relation

$$\int_{-\infty}^{\infty}d\hat P e^{i(\hat C+\hat P)(\hat z'-\hat z)}\frac{\partial}{\partial\hat P}F(\hat P)=-i(\hat z'-\hat z)(1+\hat q|\hat z'-\hat z|)e^{i\hat C(\hat z'-\hat z)-|\hat q(\hat z'-\hat z)|} \quad (8)$$

to eq. (2) generates

$$\begin{aligned}&\frac{\partial}{\partial\hat z}\tilde R(\hat z,\hat k_\perp,\hat C_{3d})\\ &=-i\int_0^{\hat z}d\hat z'\left[\tilde R(\hat z',\hat k_\perp,\hat C_{3d})+i\hat\Lambda_p^2\frac{\partial}{\partial\hat z'}\tilde R(\hat z',\hat k_\perp,\hat C_{3d})\right](\hat z'-\hat z),\\ &\times[1+\hat q(\hat z-\hat z')]e^{-(\hat q+i\hat C_{3d})(\hat z-\hat z')}+\frac{e\theta_s}{2\varepsilon_0\Gamma}\int_{-\infty}^{\infty}\tilde f_1(\vec k_\perp,\hat P,0)e^{-i(\hat C_{3d}+\hat P)\hat z}d\hat P\end{aligned} \quad (9)$$

where we define

$$\hat C_{3d}\equiv\hat C-\hat k_\perp^2. \quad (10)$$

The third derivative of eq. (9) with respects to $\hat z$ reads

$$\begin{aligned}&\tilde R^{(4)}+3(i\hat C_{3d}+\hat q)\tilde R^{(3)}+\left[\hat\Lambda_p^2+3(i\hat C_{3d}+\hat q)^2\right]\tilde R^{(2)}\\ &+\left[(i\hat C_{3d}+3\hat q)\hat\Lambda_p^2+(i\hat C_{3d}+\hat q)^3-i\right]\tilde R^{(1)}\\ &-i(i\hat C_{3d}+3\hat q)\tilde R=\frac{1}{n_0}\int_{-\infty}^{\infty}(\hat q-i\hat P)^3\tilde f_1(\hat P,0)e^{-i(\hat P+\hat C_{3d})\hat z}d\hat P\end{aligned}. \quad (11)$$

Eq. (11) is a fourth inhomogeneous ordinary differential equation (ODE) with constant coefficients and its Green function with initial condition

$$G(0)=G^{(1)}(0)=G^{(2)}(0)=G^{(3)}(0)=0 \quad (12)$$

is given by

$$G(\hat z)=\sum_{(i,j,k,l)}\frac{e^{\lambda_i\hat z}}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)(\lambda_i-\lambda_l)}, \quad (13)$$

where λ_i are solutions of the polynomial equation,

$$\begin{aligned}&\lambda^4+3(i\hat C_{3d}+\hat q)\lambda^3+\left[\hat\Lambda_p^2+3(i\hat C_{3d}+\hat q)^2\right]\lambda^2\\ &+\left[(i\hat C_{3d}+3\hat q)\hat\Lambda_p^2+(i\hat C_{3d}+\hat q)^3-i\right]\lambda=i(i\hat C_{3d}+3\hat q)\end{aligned},$$

and the summation is over the cyclic permutation of the four indices. The particular solution of eq. (11) is

$$\begin{aligned}\tilde R_p(\hat z)&=\int_0^{\hat z}G(\hat z-\hat z_1)f(\hat z_1)d\hat z_1\\ &=\frac{e\theta_s}{2\varepsilon_0\Gamma}\sum_{(i,j,k,l)}\frac{1}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)(\lambda_i-\lambda_l)}\\ &\times\int_{-\infty}^{\infty}\frac{(\hat q-i\hat P)^3\cdot\left[e^{\lambda_i\hat z}-e^{-i(\hat P+\hat C_{3d})\hat z}\right]\tilde f_1(\hat k_\perp,\hat P,0)}{\lambda_i+i(\hat P+\hat C_{3d})}d\hat P\end{aligned}. \quad (14)$$

Since eq. (14) satisfies

$$\tilde R(0)=\tilde R^{(1)}(0)=\tilde R^{(2)}(0)=\tilde R^{(3)}(0)=0, \quad (15)$$

the initial conditions are solely satisfied by the general solution, i.e.

$$\tilde R_g(\hat z)=\sum_{i=1}^{4}A_i(\hat C,\hat k_\perp)e^{\lambda_i\hat z}. \quad (16)$$

From eq. (3), the solution for radiation field is

$$\tilde E(\hat z,\hat k_\perp,C)=e^{-i\hat k_\perp^2\hat z}\left[\tilde R_g(\hat z,\hat k_\perp,C)+\tilde R_s(\hat z,\hat k_\perp,C)\right]. \quad (17)$$

The current density is related to the radiation field by [5, 6]

$$\tilde j_1(\hat z,\hat C,\hat k_\perp)=-\frac{c\Gamma}{2\pi\theta_s}\left[i\hat k_\perp^2+\frac{\partial}{\partial\hat z}\right]\tilde E. \quad (18)$$

Inserting eq. (14), eq. (16), eq. (17) into eq. (18) leads to

$$\begin{aligned}&\tilde j_1(\hat z,\hat C,\hat k_\perp)\\ &=-e^{-i\hat k_\perp^2\hat z}\left\{\frac{2c\varepsilon_0\Gamma}{\theta_s}\sum_{i=1}^{4}\lambda_iA_ie^{\lambda_i\hat z}+\sum_{(i,j,k,l)}\frac{ec}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)(\lambda_i-\lambda_l)}\right.,\\ &\left.\times\int_{-\infty}^{\infty}\frac{(\hat q-i\hat P)^3\cdot\left[\lambda_ie^{\lambda_i\hat z}+i(\hat P+\hat C_{3d})e^{-i(\hat P+\hat C_{3d})\hat z}\right]\tilde f_1(\hat k_\perp,\hat P,0)}{\lambda_i+i(\hat P+\hat C_{3d})}d\hat P\right\}\end{aligned} \quad (19)$$

where

$$\begin{pmatrix}A_1\\A_2\\A_3\\A_4\end{pmatrix}=-\frac{\theta_s}{2\varepsilon_0c\Gamma}M^{-1}\begin{pmatrix}-\frac{2\varepsilon_0c\Gamma}{\theta_s}\tilde E\\ \tilde j_1\\ \tilde j_1^{(1)}\\ \tilde j_1^{(2)}\end{pmatrix}_{\hat z=0}, \quad (20)$$

with

$$M=\begin{pmatrix}1&1&1&1\\ \lambda_1&\lambda_2&\lambda_3&\lambda_4\\ \lambda_1(\lambda_1-i\hat k_\perp^2)&\lambda_2(\lambda_2-i\hat k_\perp^2)&\lambda_3(\lambda_3-i\hat k_\perp^2)&\lambda_4(\lambda_4-i\hat k_\perp^2)\\ \lambda_1(\lambda_1-i\hat k_\perp^2)^2&\lambda_2(\lambda_2-i\hat k_\perp^2)^2&\lambda_3(\lambda_3-i\hat k_\perp^2)^2&\lambda_4(\lambda_4-i\hat k_\perp^2)^2\end{pmatrix}. \quad (21)$$

The linearized Vlasov equation for electrons in a straight field-free section reads[6]

$$\frac{\partial}{\partial\hat z}\tilde f_1+i(\hat C+\hat P)\tilde f_1=0. \quad (22)$$

Assuming the electron beam goes through a field-free straight section before entering the undulator, the initial current modulation and its derivatives can be derived from eq. (22) as

$$\left.\frac{\partial^{(n)}}{\partial\hat z^{(n)}}\tilde j(\hat z,\hat P)\right|_{\hat z=0}=\int_{-\infty}^{\infty}(-i)^n(\hat C+\hat P)^n\tilde f_1(0,\hat C,\hat P)d\hat P. \quad (23)$$

In order to proceed further, the initial phase space density modulation has to be specified.

GAUSSIAN PROFILE

For simplicity, we ignore the energy modulation and assume the initial phase space density perturbation has the following form

$$\tilde f_1(0,\hat C,\hat k_\perp,\hat P)=\frac{1}{ec}\tilde j_1(0,\hat C,\hat k_\perp)\delta(\hat P). \quad (24)$$

Inserting eq. (20), eq. (21), eq. (23) and eq. (24) into eq. (19) produces

$$\begin{aligned}\tilde j_1(\hat z,\hat C,\hat k_\perp)&=-e^{-i\hat k_\perp^2\hat z}\tilde j_1(0,\hat C,\hat k_\perp)\sum_{(i,j,k,l)}\frac{1}{(\lambda_i-\lambda_j)(\lambda_i-\lambda_k)(\lambda_i-\lambda_l)}\\ &\times\left[\lambda_i\left(B_{jkl}+\frac{\hat q^3}{\lambda_i+i\hat C_{3d}}\right)e^{\lambda_i\hat z}+\frac{i\hat q^3\hat C_{3d}}{\lambda_i+i\hat C_{3d}}e^{-i\hat C_{3d}\hat z}\right]\end{aligned} \quad (25)$$

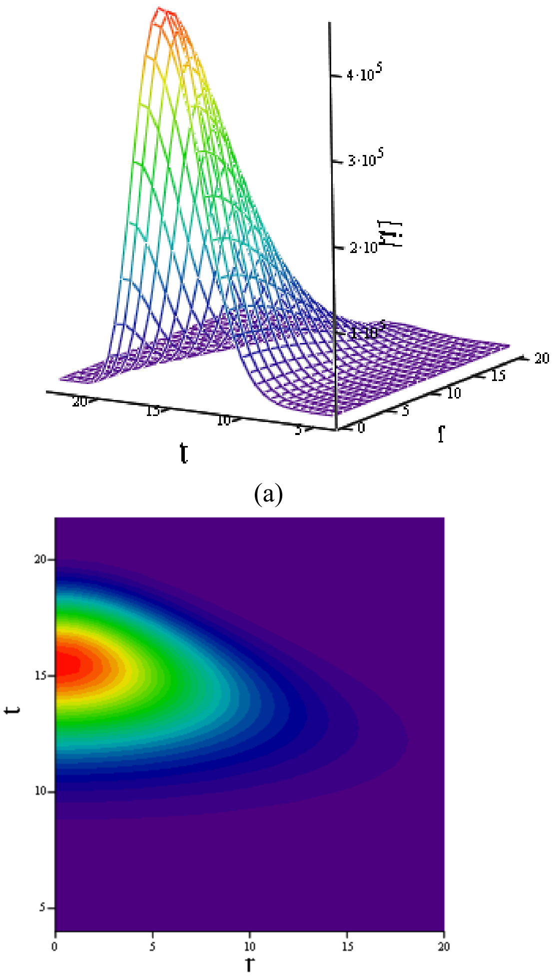

Figure 2 The amplitude of amplified current density modulation as calculated from eq. (29). The snapshot is taken at $\hat{z}=21$ with $\hat{\sigma}_t=1.37\times10^{-5}$, $\hat{\Lambda}_p=0$, $\hat{\sigma}_\perp=2.32$ and $\hat{q}=0.1$. (a) the amplitude of current modulation as a function of transverse spatial radius and longitudinal arrival time; (b) contour plot of (a).

where

$$B_{jkl}\equiv\lambda_j\lambda_k+\lambda_j\lambda_l+\lambda_k\lambda_l+i\hat{C}_{3d}\left(\lambda_j+\lambda_k+\lambda_l\right)-\hat{C}_{3d}^2 . \quad (26)$$

Assuming the initial current density perturbation has Gaussian spatial profile, i.e.

$$j_1(z,t,x,y)=\frac{ec}{2\pi\sigma_\perp^2\sqrt{2\pi}\sigma_t}e^{-\frac{r^2}{2\sigma_\perp^2}}e^{-\frac{(z-\beta ct)^2}{2\sigma_z^2}} \quad (27)$$

for $z\le0$, the Fourier components of (27) at $z=0$ is

$$\tilde{j}_1\left(0,k_x,k_y,\hat{C}\right)=\frac{e}{\beta}e^{-\frac{k_\perp^2\sigma_\perp^2}{2}}e^{\frac{-\sigma_t^2}{2}(C-k-k_w)^2}\approx e\cdot e^{-\frac{\hat{k}_\perp^2\hat{\sigma}_\perp^2}{2}}e^{\frac{-\hat{\sigma}_z^2}{2}\left(\hat{C}-\hat{k}-\hat{k}_w\right)^2} . \quad (28)$$

Inserting eq. (28) into eq. (25) and carrying out the inverse Fourier transformations yields[4]

$$\tilde{j}_1(\vec{x},t)=-\frac{ec\Gamma^3\gamma_z^4}{2\pi^2\rho}e^{-i\hat{k}_w\xi}\sum_{(i,j,k,l)}\int_{-\infty}^{\infty}d\hat{C}_{3d}e^{-i2\gamma_z^2(\hat{z}-c\hat{t})\hat{C}_{3d}}I_x\left(\hat{r},\hat{C}_{3d}\right), \times\frac{\lambda_i\left(B_{jkl}+\frac{\hat{q}^3}{\lambda_i+i\hat{C}_{3d}}\right)e^{\lambda_i\hat{z}}+\frac{i\hat{q}^3\hat{C}_{3d}}{\lambda_i+i\hat{C}_{3d}}e^{-i\hat{C}_{3d}\hat{z}}}{\left(\lambda_i-\lambda_j\right)\left(\lambda_i-\lambda_k\right)\left(\lambda_i-\lambda_l\right)} \quad (29)$$

where

$$\xi=-\left[\hat{z}+2\gamma_z^2\left(\hat{z}-c\hat{t}\right)\right], \quad (30)$$

and

$$I_x\left(\hat{r},\hat{C}_{3d}\right)\equiv\int_0^\infty e^{-\left(\frac{\hat{\sigma}_\perp^2}{2}-i\xi\right)x}e^{\frac{-\hat{\sigma}_z^2}{2}\left(x+\hat{C}_{3d}-\hat{k}-\hat{k}_w\right)^2}J_0\left(\hat{r}_\perp\sqrt{x}\right)dx . \quad (31)$$

For $\hat{\sigma}_z=0$, the integration in eq. (31) can be carried out analytically and eq. (29) becomes

$$\tilde{j}_1(\vec{x},t)=\frac{-ec\Gamma^3\gamma_z^4e^{-i\hat{k}_w\xi}}{2\pi^2\rho}\frac{e^{-\frac{1}{4}\frac{\hat{r}^2}{\hat{\sigma}_\perp^2/2-i\xi}}}{\hat{\sigma}_\perp^2/2-i\xi}\sum_{(i,j,k,l)}\int_{-\infty}^{\infty}d\hat{C}_{3d}\frac{e^{-i2\gamma_z^2(\hat{z}-c\hat{t})\hat{C}_{3d}}}{\left(\lambda_i-\lambda_j\right)\left(\lambda_i-\lambda_k\right)\left(\lambda_i-\lambda_l\right)} . \times\left[\lambda_i\cdot\left(B_{jkl}+\frac{\hat{q}^3}{\lambda_i+i\hat{C}_{3d}}\right)e^{\lambda_i\hat{z}}+\frac{i\hat{q}^3\hat{C}_{3d}}{\lambda_i+i\hat{C}_{3d}}e^{-i\hat{C}_{3d}\hat{z}}\right] \quad (32)$$

Fig. 1 shows the amplitude of the current density modulation as a function of the transverse spatial coordinate and arrival time as calculated from eq. (29). The transverse radius is in unit of $\sqrt{\rho}/\left(\sqrt{2}\gamma_z\Gamma\right)$, the longitudinal location is in unit of $1/\left(2\gamma_z^2\Gamma c\right)$, and the current density is in unit of $ec\Gamma^3\gamma_z^4/\left(2\pi^2\rho\right)$.

DISCUSSION

Eq. (29) has similar form as what have been previously obtained for current modulation due to external field excitation [5]. The contribution from the inhomogeneous driving term in eq. (11) is proportional to $\hat{q}^3$ and hence is negligible for $\hat{q}<<1$.

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