ISBN: 978-3-95450-168-7 doi:10.18429/JACoW-IPAC2015-MOPWA038

SENSITIVITY OF LINAC OPTICS TO FOCUSING AND ENERGY ERRORS

V.Balandin*, W.Decking, N.Golubeva, DESY, Hamburg, Germany

Abstract

The ability to control beam optics in the presence of such imperfections as focusing and energy gain errors is essential for a successful operation of high brightness electron linacs providing beams for free-electron lasers. We characterize the cumulative effect of these imperfections using the value of mismatch parameter calculated at the linac exit and show how it depends on the design of the focusing lattice.

INTRODUCTION

Control of the optics matching is one of the key ingredients for a successful operation of modern high brightness electron linacs providing beams for free-electron lasers (FELs). Available operational experience indicates that in order to optimize FEL signal at different wavelengths or to fine-tune the FEL wavelength, empirical adjustment of the machine parameters is often required and, therefore, the sensitivity of the beamline optics to small changes in the beam energy and in the magnet settings becomes one of the important issues which affects both, the final performance and the reproducibility of the results after breaks in operation. This fact was quickly recognized when the FLASH facility at DESY started its regular user operation in August 2005. In a little while after that the simple criteria for comparison of the optics sensitivities was introduced, new (lower sensitivity) optics for the FLASH beamline was developed and brought in operation in spring 2006, and has shown a superior performance with respect to the previous setup of the transverse focusing [1]. Later on this criteria also has been usefully adopted as a part of the optics redesign strategy during commissioning of the FERMI@Elettra FEL facility [2].

The purpose of this paper is to give some generalizations and provide details of the derivation of the optics sensitivity criteria which were missing in [1], and practical examples and discussions of the application strategies can be found in the papers [1] and [2].

DYNAMICAL VARIABLES AND TWISS PARAMETERS

We consider the linear beam dynamics with acceleration in one degree of freedom (lets say, horizontal) and use the variables $z = (x, q)^\top$ for the description of the horizontal beam oscillations. We assume that evolution of these variables along the linac is described by the linear equation

$$dz \,/\, d\tau \;=\; F(\tau)\, z, \tag{1}$$

where the independent variable τ is the longitudinal position. As concerning the physical meaning of the variables z, we do not see any particular reasons to specify it at this point, because, first, there is no single commonly accepted transverse coordinates for description of the beam dynamics in the linacs and, second, the main object of our study (the mismatch parameter) is, up to some extend, independent on the coordinate system choice.

As often, we introduce the Twiss (or Courant-Snyder) parameters using the second central moments of the particle distribution and define them as follows

$$\beta = \langle x^2 \rangle \,/\, \epsilon, \quad \alpha = -\langle xq \rangle \,/\, \epsilon, \quad \gamma = \langle q^2 \rangle \,/\, \epsilon, \tag{2}$$

where

$$\epsilon \;=\; \sqrt{\langle x^2 \rangle \langle q^2 \rangle - \langle xq \rangle^2} \tag{3}$$

is the rms emittance. With this definition, the Twiss matrix

$$\Sigma \;=\; \begin{bmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{bmatrix}, \quad \det(\Sigma) \;=\; 1 \tag{4}$$

satisfies the linear differential equation

$$d\Sigma \,/\, d\tau \;=\; JH\Sigma \,-\, \Sigma HJ, \tag{5}$$

where J is the 2×2 symplectic unit matrix and the 2×2 symmetric matrix H is defined by the expression

$$H(\tau) \;=\; \tfrac{1}{2}\mathrm{tr}[F(\tau)]\, J \,-\, JF(\tau). \tag{6}$$

The solution of Eq. (5) is given by the formula

$$\Sigma(\tau) = M(\tau)\, \Sigma(0)\, M^\top(\tau), \tag{7}$$

where the symplectic matrix M satisfies the equation

$$dM \,/\, d\tau = JHM, \quad M(0) = I, \tag{8}$$

and the rms emittance ϵ evolves according to the rule

$$\frac{\epsilon(\tau)}{\epsilon(0)} = \det\,[A(\tau)] = \exp\left(\int_0^\tau \mathrm{tr}[F(\xi)]\, d\xi \right), \tag{9}$$

where A is the fundamental matrix solution of the Eq. (1). Because the matrices M and A are connected by the relation

$$M(\tau) = \frac{1}{\sqrt{\det\,(A(\tau))}} \cdot A(\tau), \tag{10}$$

and because for an arbitrary 2×2 matrix X

$$X\, J\, X^\top = X^\top J\, X = \det(X)\, J, \tag{11}$$

the transport rule (7) can also be written as follows

$$[\Sigma(\tau) J] = A(\tau)\; [\Sigma(0) J]\; A^{-1}(\tau). \tag{12}$$

Alternatively, the matrices M and A can be expressed using Σ and ϵ, if they are known, in the familiar forms

$$M(\tau) \;=\; T^{-1}(\tau)\, R[\mu(\tau)]\, T(0), \tag{13}$$

$$A(\tau) \;=\; \sqrt{\epsilon(\tau)/\epsilon(0)} \cdot T^{-1}(\tau)\, R[\mu(\tau)]\, T(0), \tag{14}$$

where

* vladimir.balandin@desy.de

ISBN: 978-3-95450-168-7 doi:10.18429/JACoW-IPAC2015-MOPWA038

$$R(\varphi) = \begin{bmatrix} \cos(\varphi) & \sin(\varphi) \\ -\sin(\varphi) & \cos(\varphi) \end{bmatrix} \tag{15}$$

is a rotation matrix, μ is the horizontal phase advance with

$$d\mu \,/\, d\tau = h_{22} \,/\, \beta = f_{12} \,/\, \beta, \tag{16}$$

and

$$T = \begin{bmatrix} 1/\sqrt{\beta} & 0 \\ \alpha/\sqrt{\beta} & \sqrt{\beta} \end{bmatrix}, \quad \Sigma = (T^{\top} T)^{-1}. \tag{17}$$

BETATRON MISMATCH PARAMETER

Let us assume that we have two different Twiss matrices, Σ_1 and Σ_2, given at the same position in the linac and let us define a quantity called the betatron mismatch parameter as

$$m_p = (\beta_1\gamma_2 - 2\alpha_1\alpha_2 + \beta_2\gamma_1) \,/\, 2 \geq 1. \tag{18}$$

The importance of this parameter is that it stays unchanged when both Twiss matrices propagate along the same beamline and gives a numerical measure of the amplitude of the β-beat (i.e. of the ratio $\beta_2 \,/\, \beta_1$). It was introduced as a result of geometrical observation of the dynamics of the beam ellipse defined by the matrix Σ_2 in the so-called normalized coordinates connected with the matrix Σ_1 [3, 4], and later on it was shown that the inverse hyperbolic cosine of the mismatch parameter is a distance function (metric) in the space of the Twiss parameters [5].

Let us supplement the mismatch parameter m_p by the mismatch phase $\theta_{1\to2}$ defined via relations

$$\sqrt{m_p^2 - 1} \cdot \sin(2\theta_{1\to2}) = \alpha_2 - \alpha_1 \,(\beta_2 \,/\, \beta_1), \tag{19}$$

$$\sqrt{m_p^2 - 1} \cdot \cos(2\theta_{1\to2}) = (\beta_2 \,/\, \beta_1) - m_p. \tag{20}$$

Then Σ_2 can be expressed through $\Sigma_1 = (T_1^{\top} T_1)^{-1}$ as

$$\Sigma_2 = m_p \cdot \Sigma_1 + \sqrt{m_p^2 - 1} \cdot T_1^{-1} R_L(\theta_{1\to2})\, T_1^{-\top}, \tag{21}$$

where

$$R_L(\varphi) = \begin{bmatrix} \cos(2\varphi) & -\sin(2\varphi) \\ -\sin(2\varphi) & -\cos(2\varphi) \end{bmatrix} \tag{22}$$

is matrix of reflection about a line L given by equation

$$\sin(\varphi)\, x + \cos(\varphi)\, q = 0. \tag{23}$$

Note that the matrix R_L can be represented in the form

$$R_L(\varphi) = R(\varphi)\, R_L(0)\, R^{\top}(\varphi), \tag{24}$$

where $R_L(0)$ is reflection about the x-axis.

Let us assume that the representation (21) was calculated for $\tau = 0$. Then, using (13) written for $\Sigma = \Sigma_1$, one obtains

$$\Sigma_2(\tau) = m_p \cdot \Sigma_1(\tau)$$

$$+ \sqrt{m_p^2 - 1} \cdot T_1^{-1}(\tau) R_L \left[\mu_1(\tau) + \theta_{1\to2}(0)\right]\, T_1^{-\top}(\tau), \tag{25}$$

which is the matrix form of the well-known rule that the β-beat oscillates at twice the betatron frequency.

Note, for completeness, that the invariance of m_p during propagation along the linac easily follows from the transport rules (7) and (12) and the fact that the definition (18) can be equivalently written in the following forms

$$m_p = \tfrac{1}{2}\,\mathrm{tr}\left[\Sigma_2\, \Sigma_1^{-1}\right] = -\tfrac{1}{2}\,\mathrm{tr}\left[\Sigma_2\, J\, \Sigma_1\, J\right]. \tag{26}$$

Invariance under Linear Change of Variables

Let us introduce new dynamical variables $\tilde{z}$ with the help of the linear non-autonomous coordinate transformation

$$\tilde{z}(\tau) = C(\tau)\, z(\tau), \quad \det\left[C(\tau)\right] \neq 0. \tag{27}$$

Then the following relations hold

$$\tilde{\Sigma}(\tau) = \frac{1}{|\det\left[C(\tau)\right]|} \cdot C(\tau)\, \Sigma(\tau)\, C^{\top}(\tau), \tag{28}$$

$$\tilde{\epsilon}(\tau) = |\det\left[C(\tau)\right]| \cdot \epsilon(\tau). \tag{29}$$

One sees that not only the Twiss parameters, but, in general, the rms emittance also change. Nevertheless, if one calculates mismatch between the two Twiss matrices in the old and in the new variables, one obtains that it stays unchanged, i.e. $\tilde{m}_p(\tau) \equiv m_p(\tau)$. It means that for the estimation of the mismatch parameter the choice of the variables does not play essential role as far as transition between coordinate systems of interest can be approximated by means of a linear non-autonomous transformation.

PERTURBATION EXPANSION OF THE MISMATCH PARAMETER

Let us introduce an artificial small parameter $\varkappa$ so that the value $\varkappa = 0$ will indicate the beamline with the perfect (design) settings of acceleration and focusing, and a nonzero $\varkappa$ will stand for the motion through the beamline perturbed by different imperfections. We will characterize the effect of these imperfections using the value of the mismatch calculated between the two Twiss matrices, one (Σ_0) passing through the ideal beamline and other ($\Sigma_\varkappa$) moving through the beamline with errors. According to Eqs. (26) and (7)

$$m_p(\tau) = -\tfrac{1}{2}\,\mathrm{tr}\left[\Sigma_\varkappa(\tau)\, J\, \Sigma_0(\tau)\, J\right]$$

$$= \tfrac{1}{2}\,\mathrm{tr}\left[Z_\varkappa(\tau)\, \Sigma_\varkappa(0)\, Z_\varkappa^{\top}(\tau)\, \Sigma_0(0)\right], \tag{30}$$

where

$$Z_\varkappa(\tau) = M_0^{\top}(\tau)\, J\, M_\varkappa(\tau) = J\, M_0^{-1}(\tau)\, M_\varkappa(\tau), \tag{31}$$

and, in the next step, we would like to find an analytical formula for the lowest order terms with respect to $\varkappa$ when the right hand side of Eq. (30) is expanded in Taylor series. As it turns out, in order to do this, the knowledge of the matrices $M_\varkappa$ and $H_\varkappa$ in Eq. (8) up to the second order is required. So, let us assume that

$$H_\varkappa =_2 H_0 + \varkappa \tilde{H}_1 + \varkappa^2 \tilde{H}_2, \tag{32}$$

where the symbol $=_2$ denotes equality up to second order (inclusive) with respect to the parameter $\varkappa$. Applying now standard technique of the perturbation theory, one obtains after lengthy but more or less straightforward manipulations

$$m_p(\tau) =_2 m_p(0)\left[1 + D_1(\tau)\right] + \sqrt{m_p^2(0) - 1}\, D_2(\tau), \tag{33}$$

which is the desired result. In this formula

$$D_1(\tau) = \tfrac{1}{2}\varkappa^2 \left\{\mathrm{tr}^2\left[\Lambda_1(\tau)\right] - 4\det\left[\Lambda_1(\tau)\right]\right\}, \tag{34}$$

$$D_2(\tau) = \varkappa\, \mathrm{tr}\left\{\Lambda_1(\tau) R_L\left[\theta_{0\to\varkappa}(0) - \pi/4\right]\right\}$$

$$+ \tfrac{1}{2}\varkappa^2\, \mathrm{tr}\left[\Lambda_1(\tau)\right] \cdot \mathrm{tr}\left\{\Lambda_1(\tau) R_L\left[\theta_{0\to\varkappa}(0)\right]\right\}$$

$$+ \varkappa^2\, \mathrm{tr}\left\{\Lambda_2(\tau) R_L\left[\theta_{0\to\varkappa}(0) - \pi/4\right]\right\}, \tag{35}$$

ISBN: 978-3-95450-168-7 doi:10.18429/JACoW-IPAC2015-MOPWA038

and the symmetric matrices $\Lambda_1(\tau)$ and $\Lambda_2(\tau)$ are to be found by direct integration of the differential equations

$$d\Lambda_1 / d\tau = F^\top \tilde{H}_1 F, \quad \Lambda_1(0) = 0, \tag{36}$$

$$d\Lambda_2 / d\tau = F^\top \tilde{H}_2 F + \frac{1}{2}\left[\left(\frac{d\Lambda_1}{d\tau} J \Lambda_1\right) + \left(\frac{d\Lambda_1}{d\tau} J \Lambda_1\right)^\top\right], \quad \Lambda_2(0) = 0, \tag{37}$$

where

$$F(\tau) = T_0^{-1}(\tau) R[\mu_0(\tau)]. \tag{38}$$

SENSITIVITY TO ERRORS

The formula (33) contains only the lowest order terms with respect to $\varkappa$, but no assumptions about $m_p(0)$ were made during its derivation. It can be arbitrarily large, which essentially complicates Eq. (33) due to described by the function D_2 possibility for the initial mismatch to interact with the lattice imperfections. But it makes no sense to consider arbitrary entrance mismatches in the development of the some kind of the simplified optics sensitivity criteria, because the optics of the single pass beamline is by itself a function of the initial conditions. So let us assume that either $\Sigma_\varkappa(0) = \Sigma_0(0)$, or the mismatch between them is small. In the latter case two terms of the order $\varkappa^2$ in (35) can be omitted, but it is not all. Due to the inequality

$$|\mathrm{tr}[\Lambda_1 R_L(\psi)]|^2 \leq \mathrm{tr}^2(\Lambda_1) - 4\det(\Lambda_1), \quad \forall \psi, \tag{39}$$

the remaining in D_2 term will be reduced simultaneously with the minimization of the function D_1. It motivates us in the following to consider only the case when $\Sigma_\varkappa(0) = \Sigma_0(0)$ and, for the mismatch development, to use the formula

$$m_p(\tau) \approx 1 + D_1(\tau), \tag{40}$$

where

$$D_1(\tau) = \frac{1}{2}\left(\int_0^\tau [U\cos(2\mu_0) + V\sin(2\mu_0)]\, d\xi\right)^2 + \frac{1}{2}\left(\int_0^\tau [U\sin(2\mu_0) - V\cos(2\mu_0)]\, d\xi\right)^2, \tag{41}$$

$$U = \beta_0 \tilde{h}_{11} - 2\alpha_0 \tilde{h}_{12} + (\alpha_0^2 - 1)\tilde{h}_{22} / \beta_0, \tag{42}$$

$$V = 2\left(\alpha_0 \tilde{h}_{22} / \beta_0 - \tilde{h}_{12}\right), \tag{43}$$

β_0, α_0, and μ_0 are the design Twiss parameters, and we have hidden the artificial parameter $\varkappa$ into the elements $\tilde{h}_{ij}$ of the matrix $H_\varkappa - H_0$,

Let us turn our attention to the situation when focusing and acceleration is provided by quadrupoles and rotationally symmetric cavities, and let us take as variables z the particle horizontal position and the particle mechanical momentum scaled with the kinetic momentum p_0 of the ideal reference particle. In these variables the matrix $H_\varkappa$ can be expressed using the speed of light approximation as follows

$$H_\varkappa = \begin{bmatrix} k_\varkappa + \frac{1}{2p_0}\frac{d^2 p_\varkappa}{d\tau^2} & \frac{1}{2p_0}\frac{dp_0}{d\tau} \\ \frac{1}{2p_0}\frac{dp_0}{d\tau} & \frac{p_0}{p_\varkappa} \end{bmatrix}. \tag{44}$$

Here $k_\varkappa = (e / p_0)\, b_\varkappa$ is quadrupole coefficient, $b_\varkappa$ is quadrupole gradient, e is particle charge, and $p_\varkappa$ representing chromatic effects and RF focusing satisfies

$$dp_\varkappa / d\tau = (e / c)\, E_\varkappa[\tau / c + t_\varkappa(0), \tau], \tag{45}$$

where c is velocity of light, $E_\varkappa = E_\varkappa(t, \tau)$ is on-axis component of the longitudinal electric field, and $t_\varkappa(0)$ is laboratory time which particle has at the starting position $\tau = 0$. Because for the matrix (44) the element $\tilde{h}_{12}$ is equal to zero, Eq. (41) can be rewritten in the following useful form

$$D_1(\tau) = \frac{1}{2}\left(\int_0^\tau \left[\beta_0 \cos(2\mu_0)\tilde{h}_{11} - 2\frac{d\zeta_0}{d\xi}\tilde{h}_{22}\right] d\xi\right)^2 + \frac{1}{2}\left(\int_0^\tau \left[\beta_0 \sin(2\mu_0)\tilde{h}_{11} - 2\frac{d\eta_0}{d\xi}\tilde{h}_{22}\right] d\xi\right)^2, \tag{46}$$

where the functions ζ_0 and η_0 are the apochromaticities [6],

$$\tilde{h}_{11} = k_\varkappa - k_0 + \frac{1}{2p_0}\frac{d^2(p_\varkappa - p_0)}{d\tau^2} \tag{47}$$

contains quadrupole gradient and RF focusing errors, and

$$\tilde{h}_{22} = -(p_\varkappa - p_0) / p_\varkappa \tag{48}$$

characterizes influence of chromatic effects. In order to separate the influence of focusing and chromatic imperfections, let us estimate the function D_1 from above as follows

$$D_1(\tau) \leq D_{1A}(\tau) + D_{1B}(\tau), \tag{49}$$

where

$$D_{1A} = \left(\int_0^\tau \beta_0 \cos(2\mu_0)\tilde{h}_{11} d\xi\right)^2 + \left(\int_0^\tau \beta_0 \cos(2\mu_0)\tilde{h}_{11} d\xi\right)^2,$$

$$D_{1B} = \left(\int_0^\tau 2\frac{d\zeta_0}{d\xi}\tilde{h}_{22} d\xi\right)^2 + \left(\int_0^\tau 2\frac{d\eta_0}{d\xi}\tilde{h}_{22} d\xi\right)^2.$$

Let us further assume that the errors in the focusing are proportional to the focusing itself, i.e. that

$$\tilde{h}_{11} = \left(k_0 + \frac{1}{2p_0}\frac{d^2 p_0}{d\tau^2}\right) \cdot \tilde{\delta}, \tag{50}$$

and let us estimate D_{1A} not using the Cauchy-Bunyakovsky inequality as it was done in [1], but simply as

$$D_{1A} \leq 2 \max_\tau \tilde{\delta}^2 \cdot \left(\int_0^\tau \left|k_0 + \frac{1}{2p_0}\frac{d^2 p_0}{d\xi^2}\right| \beta_0\, d\xi\right)^2. \tag{51}$$

Then the integral in the parentheses in the right hand side of (51) can be taken as a rough criteria of the optics sensitivity to the focusing errors. By analogy, and using that

$$(d\zeta_0 / d\tau)^2 + (d\eta_0 / d\tau)^2 = (\gamma_0 / 2)^2, \tag{52}$$

one obtains the upper estimate of D_{1B} as follows

$$D_{1B} \leq 2 \max_\tau \tilde{h}_{22}^2 \cdot \left(\int_0^\tau \gamma_0\, d\xi\right)^2, \tag{53}$$

and, correspondingly, the integral of γ_0 (which is equal to two times the absolute value of the beamline chromaticity) can be taken as the sensitivity criteria to the chromatic errors. But it seems that due to relation

$$\gamma_0 = \left(k_0 + \frac{1}{2p_0}\frac{d^2 p_0}{d\xi^2}\right)\beta_0 + \frac{d\alpha_0}{d\tau}, \tag{54}$$

this additional criteria (though important by itself) can be skipped in the framework of this paper.

ISBN: 978-3-95450-168-7 doi:10.18429/JACoW-IPAC2015-MOPWA038

REFERENCES

[1] V.Balandin, N.Golubeva, Low sensitivity option for transverse optics of the FLASH linac at DESY, PAC'09, Vancouver, BC, Canada, May 2009, TH6PFP031.

[2] S.Di Mitri, M.Cornacchia, Merit functions for the linac optics design for colliders and light sources, NIM A 735 (2014).

[3] P.Bambade, Betatron Phase-Space Diagnostics in a FODO Array, SLAC-CN-367, 1988.

[4] M.Sands, A Beta Mismatch Parameter, SLAC-AP-085, 1991.

[5] V.Balandin, N.Golubeva, Dynamics of Twiss parameters from the geometrical viewpoint, IPAC'14, Dresden, Germany, June 2014, TUPRO048.

[6] V.Balandin et al., Apochromatic Beam Transport in Drift-Quadrupole Systems, IPAC'10, Kyoto, Japan, May 2010, THPD083.